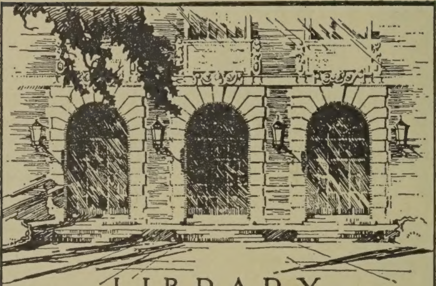
HANUS

THEORY OF DETERMINANTS



AN

ELEMENTARY TREATISE

ON THE

THEORY OF DETERMINANTS.

A TEXT-BOOK FOR COLLEGES.

BY

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PREFACE.

The importance of a knowledge of Determinants to all who extend their reading beyond the elements of mathematics, and the fact that most modern writers employ the determinant notation, have led to the belief that an American work on Determinants might satisfy a growing demand.

This is a text-book, and not an exhaustive treatise. Enough is given, however, to enable the student to use the determinant notation with ease, and to enable him to pursue his further reading in the modern higher mathematics with pleasure and profit.

The book is written with reference to the wants of the private student as well as to the needs of the class-room. The subject is at first presented with great simplicity. As the student advances, less attention is given to details. More than half the volume is devoted to applications and special forms, that the reader may get *some* notion of the power and utility of determinants as instruments of research.

Throughout the work care has been taken to show how each new concept has been evolved naturally; and, whenever it is thought advisable, a special case precedes the general discussion.

The work has been written in the far West, where contact with others in the same field was practically impossible. I

shall therefore be grateful for any notification of errors that may have escaped detection.

My thanks are due to Messrs. J. S. Cushing & Co., of Boston, for great care and patience manifested in the preparation of the plates.

Among the works consulted most assistance has been derived from the following. All the works named have been used freely.

Matzka. — Grundzüge der systematischen Einführung und Begründung der Lehre der Determinanten.

Baltzer. — Theorie und Anwendung der Determinanten (Fünfte Auflage).

Günther. — Lehrbuch der Determinanten-Theorie (Zweite Auflage).

Diekmann. — Einleitung in die Lehre von den Determinanten und ihrer Anwendung auf, etc.

Dostor. — Éléments de la Theorie des Déterminants avec Applications, etc. (Deuxième edition).

Hoüel. — Cours de Calcul Infinitésimal.

Scott. — A Treatise on the Theory of Determinants and their Applications, etc.

Burnside and Panton. — The Theory of Equations, with an Introduction, etc.

Muir. — A Treatise on the Theory of Determinants.

I am especially indebted to the last two works for many examples.

PAUL H. HANUS.

Boulder, Col., May, 1886.

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THEORY OF DETERMINANTS.

CHAPTER I.

PRELIMINARY NOTIONS AND DEFINITIONS.

1. The first notion of *Determinants* we owe to Leibnitz, who, in his attempts to simplify the expressions arising in the elimination of the unknown quantities from a set of linear equations, employed symbols nearly identical with our present determinant notation. In a letter dated April 28, 1693, Leibnitz communicates his discovery to L'Hospital; and later, in another letter, expresses the conviction that the functions will develop remarkable and very important properties, — a conviction which time has abundantly verified. Leibnitz, however, never pursued the subject himself, and his discovery lay dormant till the middle of the eighteenth century.

In 1750 the celebrated geometer, Gabriel Cramer, rediscovered determinants while working upon the analysis of curves. During the course of his investigations, Cramer had to solve sets of linear equations, and naturally encountered the same functions that had attracted the attention of Leibnitz.* To Cramer is due the general rule for the solution of n simultaneous linear equations (non-homogeneous), containing as many unknown quantities.

This rule was inferred *without proof* from the form of the values of the unknown quantities obtained in solving sets of two and three equations.

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* The particular problem which led to Cramer's discovery of determinants appears to have been: To pass a curve of the vth order through any $\frac{v^2}{2}+\frac{3v}{2}$ given points.

Since the time of Cramer important advances have been made. The names of many celebrated mathematicians appear in the list of those who aided the evolution of a theory of determinants. Prominent among these are Vandermonde and Gauss. From Gauss the name "determinant" instead of "resultant" was adopted by Cauchy. Cauchy and Jacobi are perhaps to be considered as the greatest among those who first developed the subject. The monograph of Jacobi, published in 1841,* established the foundation of a treatise on the theory of determinants; and his own writings, as well as the works of many eminent mathematicians during the past fifty years, attest the wonderful power of determinants as instruments of mathematical investigation, and the fruitfulness of the functions themselves.

2. The most natural way of approaching the theory of determinants would be along the line of development. This is accordingly our purpose. Owing to peculiar difficulties attending this mode of procedure, we can however only employ this method at the outset, and must soon adopt a presentation better suited to the further unfolding of the subject, and free from the peculiar difficulties alluded to.

Determinants of the second, third, and fourth order.

3. Consider the set of four simultaneous linear equations:—

$$\left.\begin{array}{ll} (1) & a_1x + b_1y + c_1z + d_1t = m_1 \\ (2) & a_2x + b_2y + c_2z + d_2t = m_2 \\ (3) & a_3x + b_3y + c_3z + d_3t = m_3 \\ (4) & a_4x + b_4y + c_4z + d_4t = m_4 \end{array}\right\} \text{I.}$$

Here it will be convenient to eliminate the unknown quantities in a uniform manner, as follows: in each set of equations to be obtained, (2) will be multiplied by the coefficient of the unknown in (1) that is to be eliminated, and (1) by the corresponding coefficient in (2); (3) will be multiplied by the

* *De Formatione et Proprietatibus Determinantium.*

coefficient of the unknown under consideration in (2), and (2) by the corresponding coefficient in (3); and so on through the set. Having thus made the coefficients of one of the unknowns, x, say, the same in all the equations, we will then eliminate x by subtracting (1) from (2), (2) from (3), etc. We shall find in performing these operations that the coefficients of the unknown quantities and the absolute term after each elimination are functions of a particular form, and subject to the same law of formation,—that these functions are, in fact, DETERMINANTS.

Eliminating x in set I. as directed, we have

$$\left.\begin{array}{ll}(1) & (a_1b_2-a_2b_1)y+(a_1c_2-a_2c_1)z+(a_1d_2-a_2d_1)t=a_1m_2-a_2m_1 \\ (2) & (a_2b_3-a_3b_2)y+(a_2c_3-a_3c_2)z+(a_2d_3-a_3d_2)t=a_2m_3-a_3m_2 \\ (3) & (a_3b_4-a_4b_3)y+(a_3c_4-a_4c_3)z+(a_3d_4-a_4d_3)t=a_3m_4-a_4m_3\end{array}\right\}\text{II.}$$

4. Examining these binomial coefficients, we see that each contains one positive and one negative term, and involves four quantities, viz., a_1, a_2, b_1, b_2; or a_2, a_3, c_2, c_3, etc. It will also be noticed that each term never contains more than one a (coefficient of x), or b (coefficient of y), or c (coefficient of z), etc., but that each term does contain all the subscripts that occur in the binomial. Finally, the terms in which the subscripts occur in their natural order are positive, while in the negative terms there is an inversion of the natural order in the subscripts, *i.e.*, a_3c_4 is $+$, but a_4c_3 is $-$. Such binomials are determinants of the *second order*.* (The order of a determinant is determined by the number of factors in each term.) It has been agreed to denote them, following Laplace, by writing the letters involved in regular succession, affecting each with the subscripts in order, and enclosing the whole expression within parentheses, thus: $(a_1b_2)\equiv a_1b_2-a_2b_1$; $(a_2c_3)\equiv a_2c_3-a_3c_2$, etc.

Introducing this notation, set II. becomes

$$\left.\begin{array}{ll}(1) & (a_1b_2)\,y+(a_1c_2)\,z+(a_1d_2)\,t=(a_1m_2) \\ (2) & (a_2b_3)\,y+(a_2c_3)\,z+(a_2d_3)\,t=(a_2m_3) \\ (3) & (a_3b_4)\,y+(a_3c_4)\,z+(a_3d_4)\,t=(a_3m_4)\end{array}\right\}\text{III.}$$

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* The general definition of a determinant is given in **17**, Chap. II.

5. If we now eliminate y, according to the directions given in **3**, we have

$$\left.\begin{array}{ll}(1) & [(a_1b_2)(a_2c_3)-(a_2b_3)(a_1c_2)]z+[(a_1b_2)(a_2d_3) \\ & \quad -(a_2b_3)(a_1d_2)]t=(a_1b_2)(a_2m_3)-(a_2b_3)(a_1m_2) \\ (2) & [(a_2b_3)(a_3c_4)-(a_3b_4)(a_2c_3)]z+[(a_2b_3)(a_3d_4) \\ & \quad -(a_3b_4)(a_2d_3)]t=(a_2b_3)(a_3m_4)-(a_3b_4)(a_2m_3)\end{array}\right\}\text{IV.}$$

Examining the binomial coefficients of the unknowns, and the absolute terms in set IV, we see at once that they are of the same form; and if we can simplify any one of them and discover the law of formation, we have them all. For this purpose let us expand the coefficient of z, putting, for shortness, this coefficient equal to C. Then, by the definition in **4**,

$$\begin{aligned}C &\equiv (a_1b_2)(a_2c_3)-(a_2b_3)(a_1c_2) \\ &= (a_1b_2)(a_2c_3-a_3c_2)-(a_2b_3)(a_1c_2-a_2c_1) \\ &= a_2[(a_1b_2)c_3+(a_2b_3)c_1]-c_2[(a_1b_2)a_3+(a_2b_3)a_1].\end{aligned}$$

The last binomial,

$$\begin{aligned}(a_1b_2)a_3+(a_2b_3)a_1 &= (a_1b_2-a_2b_1)a_3+(a_2b_3-a_3b_2)a_1 \\ &= a_2(a_1b_3-a_3b_1)=a_2(a_1b_3).\end{aligned}$$

$$\begin{aligned}\therefore C &= a_2[(a_1b_2)c_3-(a_1b_3)c_2+(a_2b_3)c_1] \\ &= a_2[a_1b_2c_3-a_2b_1c_3-a_1b_3c_2+a_3b_1c_2+a_2b_3c_1-a_3b_2c_1].\end{aligned}$$

Here the quantity within brackets consists of $2\cdot 3=6$ terms, *i.e.*, of as many terms as there are permutations of the subscripts 1, 2, 3. Three of the terms are positive and as many are negative. The quantity involves the $3^2=9$ quantities a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3.

No term involves more than one a, or b, or c, but does contain all of the subscripts 1, 2, 3, each term containing a different permutation of these numbers. Finally, as before, we notice that those terms in which the subscripts occur in their natural order, or in which there is an even number of inversions* of

* In a series of integers which are all different there is said to be an inversion of order when a greater number precedes a less. Thus in 13452 there are three inversions, in 21354 there are two inversions, etc.

order, are positive, while those terms are negative in which the number of inversions of order of the subscripts is odd. Such a function is a determinant of the *third order*. A determinant in which the quantities are those of C is denoted by $(a_1 b_2 c_3)$. We therefore have $C \equiv a_2(a_1 b_2 c_3)$. It must be carefully noticed that the equation

$$\begin{aligned}(a_1 b_2 c_3) &= (a_1 b_2)\, c_3 - (a_1 b_3)\, c_2 + (a_2 b_3)\, c_1 \\ &= a_1 b_2 c_3 - a_2 b_1 c_3 - a_1 b_3 c_2 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1\end{aligned}$$

gives the expansion of a determinant of the third order.

Employing the notation just explained, the coefficient of t in (1) is evidently $a_2(a_1 b_2 d_3)$, and the absolute term is $a_2(a_1 b_2 m_3)$. The coefficients and the absolute term of (2) will obviously be $a_3(a_2 b_3 c_4)$, $a_3(a_2 b_3 d_4)$, $a_3(a_2 b_3 m_4)$, in order.

Introducing this notation into set IV, and dividing (1) and (2) by a_2 and a_3 respectively, we have

$$\left.\begin{aligned}&(1)\quad (a_1 b_2 c_3)\, z + (a_1 b_2 d_3)\, t = (a_1 b_2 m_3) \\ &(2)\quad (a_2 b_3 c_4)\, z + (a_2 b_3 d_4)\, t = (a_2 b_3 m_4)\end{aligned}\right\} \text{V.}$$

6.* If we now eliminate z in the same manner as heretofore, we have

$$\left.\begin{aligned}&[(a_1 b_2 c_3)\,(a_2 b_3 d_4) - (a_2 b_3 c_4)\,(a_1 b_2 d_3)]\, t \\ &\qquad = (a_1 b_2 c_3)\,(a_2 b_3 m_4) - (a_2 b_3 c_4)\,(a_1 b_2 m_3)\end{aligned}\right\} \text{VI.}$$

The preceding results naturally imply a simplification and law of formation to be discovered in the coefficient of t and the absolute term of VI.

To simplify the coefficient of t, which for shortness we will call C, as before, we proceed as follows:

$$\begin{aligned}C &\equiv (a_1 b_2 c_3)\,(a_2 b_3 d_4) - (a_2 b_3 c_4)\,(a_1 b_2 d_3) \\ &= (a_1 b_2 c_3)\,[(a_2 b_3)\, d_4 - (a_2 b_4)\, d_3 + (a_3 b_4)\, d_2] \\ &\qquad - (a_2 b_3 c_4)\,[(a_1 b_2)\, d_3 - (a_1 b_3)\, d_2 + (a_2 b_3)\, d_1] \\ &= (a_2 b_3)\,[(a_1 b_2 c_3)\, d_4 - (a_2 b_3 c_4)\, d_1] - D d_3 + D_1 d_2;\end{aligned}$$

* **6** may be omitted on first reading, if thought best.

in which

$$D \equiv (a_1b_2c_3)\,(a_2b_4) + (a_2b_3c_4)\,(a_1b_2), \text{ and}$$
$$D_1 \equiv (a_1b_2c_3)\,(a_3b_4) + (a_2b_3c_4)\,(a_1b_3).$$

Now, by **5**, $(a_1b_2c_3) = (a_1b_2)\,c_3 - (a_1b_3)\,c_2 + (a_2b_3)\,c_1$;

and $\qquad (a_2b_3c_4) = (a_2b_3)\,c_4 - (a_2b_4)\,c_3 + (a_3b_4)\,c_2.$

Substituting,

$$\begin{aligned} D &= (a_2b_4)\,[(a_1b_2)\,c_3 - (a_1b_3)\,c_2 + (a_2b_3)\,c_1] + (a_1b_2)\,[(a_2b_3)\,c_4 \\ &\qquad - (a_2b_4)\,c_3 + (a_3b_4)\,c_2] \\ &= (a_2b_3)\,[(a_2b_4)\,c_1 + (a_1b_2)\,c_4] - [(a_2b_4)\,(a_1b_3) \\ &\qquad - (a_1b_2)\,(a_3b_4)]\,c_2. \end{aligned}$$

The second binomial, $(a_2b_4)\,(a_1b_3) - (a_1b_2)\,(a_3b_4)$

$$\begin{aligned} &= (a_2b_4 - a_4b_2)\,(a_1b_3) - (a_3b_4 - a_4b_3)\,(a_1b_2) \\ &= b_4[(a_1b_3)\,a_2 - (a_1b_2)\,a_3] - a_4[(a_1b_3)\,b_2 - (a_1b_2)\,b_3] \\ &= b_4[(a_1b_3 - a_3b_1)\,a_2 - (a_1b_2 - a_2b_1)\,a_3] \\ &\qquad - a_4[(a_1b_3 - a_3b_1)\,b_2 - (a_1b_2 - a_2b_1)\,b_3] \\ &= a_1b_4\,(a_2b_3 - a_3b_2) - a_4b_1\,(a_2b_3 - a_3b_2) \\ &= (a_1b_4)\,(a_2b_3) \quad \ldots\ldots\ldots\ldots \quad (\mathbf{K}) \end{aligned}$$

$$\therefore D = (a_2b_3)\,[(a_1b_2)\,c_4 - (a_1b_4)\,c_2 + (a_2b_4)\,c_1].$$

Substituting the expansions of $(a_1b_2c_3)$ and of $(a_2b_3c_4)$ in D_1 we have

$$\begin{aligned} D_1 &= (a_3b_4)\,[(a_1b_2)\,c_3 - (a_1b_3)\,c_2 + (a_2b_3)\,c_1] + (a_1b_3)\,[(a_2b_3)\,c_4 \\ &\qquad - (a_2b_4)\,c_3 + (a_3b_4)\,c_2] \\ &= (a_2b_3)[(a_3b_4)c_1 + (a_1b_3)c_4] - [(a_1b_3)\,(a_2b_4) - (a_1b_2)\,(a_3b_4)]c_3. \end{aligned}$$

Here we notice that the binomial factor of the second term is the same as the binomial factor in the last term of D: hence equation (**K**) above, is $(a_2b_3)\,(a_1b_4)$.

$$\therefore D_1 = (a_2b_3)\,[(a_1b_3)c_4 - (a_1b_4)c_3 + (a_3b_4)c_1].$$

Substituting the values of D and D_1 just obtained, in C, we have

$$C \equiv (a_2b_3)\left[(a_1b_2c_3)d_4 - (a_2b_3c_4)d_1 - \{(a_1b_2)c_4 - (a_1b_4)c_2 + (a_2b_4)c_1\}d_3 + \{(a_1b_3)c_4 - (a_1b_4)c_3 + (a_3b_4)c_1\}d_2\right]$$
$$= (a_2b_3)\left[(a_1b_2c_3)d_4 - (a_1b_2c_4)d_3 + (a_1b_3c_4)d_2 - (a_2b_3c_4)d_1\right].$$

From this value of C the absolute term of VI is obviously

$$(a_2b_3)\left[(a_1b_2c_3)m_4 - (a_1b_2c_4)m_3 + (a_1b_3c_4)m_2 - (a_2b_3c_4)m_1\right].$$

Now the quantity within brackets in C (and in the absolute term) of VI is here seen to be composed of four terms, each of which contains a factor which is a determinant of the third order. We shall presently show that this quantity is a determinant of the *fourth order*, and will therefore write, in accordance with the notation already exemplified, for determinants of lower orders:

$$(a_1b_2c_3)d_4 - (a_1b_2c_4)d_3 + (a_1b_3c_4)d_2 - (a_2b_3c_4)d_1 \equiv (a_1b_2c_3d_4) \qquad (\mathbf{R}).$$

Now, **5**,
$$(a_1b_2c_3) = (a_1b_2)c_3 - (a_1b_3)c_2 + (a_2b_3)c_1;$$
$$(a_1b_2c_4) = (a_1b_2)c_4 - (a_1b_4)c_2 + (a_2b_4)c_1;$$
$$(a_1b_3c_4) = (a_1b_3)c_4 - (a_1b_4)c_3 + (a_3b_4)c_1;$$
$$(a_2b_3c_4) = (a_2b_3)c_4 - (a_2b_4)c_3 + (a_3b_4)c_2.$$

Expanding the determinants of the second order in the second members of these equations according to **4**, and substituting in equation (**R**), there results:

$$\begin{aligned}(a_1b_2c_3d_4) \equiv\ & a_1b_2c_3d_4 - a_2b_1c_3d_4 - a_1b_3c_2d_4 + a_3b_1c_2d_4 + a_2b_3c_1d_4 - a_3b_2c_1d_4 \\ & - a_1b_2c_4d_3 + a_2b_1c_4d_3 + a_1b_4c_2d_3 - a_4b_1c_2d_3 - a_2b_4c_1d_3 + a_4b_2c_1d_3 \\ & + a_1b_3c_4d_2 - a_3b_1c_4d_2 - a_1b_4c_3d_2 + a_4b_1c_3d_2 + a_3b_4c_1d_2 - a_4b_3c_1d_2 \\ & - a_2b_3c_4d_1 + a_3b_2c_4d_1 + a_2b_4c_3d_1 - a_4b_2c_3d_1 - a_3b_4c_2d_1 + a_4b_3c_2d_1.\end{aligned}$$

This expansion contains $4\cdot3\cdot2 = 24$ terms, involving $4^2 = 16$ quantities. Each term contains only one a (coefficient of x), one b (coefficient of y), one c (coefficient of z), one d (coefficient of t), and contains all the subscripts; a different permutation of the subscripts belonging to each term. As before, we find that the number of inversions of order of the subscripts is an even number in the positive terms, and is an odd number in

the negative terms. Moreover, the number of terms is exactly the number of permutations of the first four natural numbers. Such a function is a determinant of the *fourth order*, and is accordingly designated by $(a_1 b_2 c_3 d_4)$. Introducing this notation, and dividing by $(a_2 b_3)$, equation VI becomes

$$(a_1 b_2 c_3 d_4)t = (a_1 b_2 c_3 m_4). \quad \text{VII.}$$

It is to be noticed that equation (**R**) of the present article gives the expansion of a determinant of the fourth order.

7. We have now shown how determinants of the second, third, and fourth orders arise in the solution of simple simultaneous equations. From the reductions of **6**, it is obvious that to continue the present method would very soon imply difficulties in the simplifications practically insurmountable when we attempt to produce determinants of the higher orders. For determinants of the fifth order, the process of reduction would be found very tedious. Hence, to investigate the properties of determinants of the nth order, we are forced to take a new starting-point; and in Chapter II. we proceed upon a plan somewhat different from that hitherto adopted.

Values of the Unknown Quantities.

8. From equation VII, **6**, $t = \frac{(a_1 b_2 c_3 m_4)}{(a_1 b_2 c_3 d_4)}$. Had the equations of set I been so arranged that z should be the last unknown in each equation, we would evidently have $z = \frac{(a_1 b_2 d_3 m_4)}{(a_1 b_2 d_3 c_4)}$. In the same way, $y = \frac{(a_1 d_2 c_3 m_4)}{(a_1 d_2 c_3 b_4)}$; $x = \frac{(d_1 b_2 c_3 m_4)}{(d_1 b_2 c_3 a_4)}$.

9. Among the many properties of determinants to be established, we may here produce the following theorem, which is among the most important of the elementary theorems in the subject:

The interchange of two letters, or of two subscripts, the others remaining undisturbed, changes the sign but not the magnitude of a determinant.

1st. For determinants of the second order.

(**a**) The interchange of two *letters*.

$(a_1 b_2) \equiv a_1 b_2 - a_2 b_1$. In this, if we interchange a and b, the second member becomes

$$b_1 a_2 - b_2 a_1 = -(a_1 b_2 - a_2 b_1) \therefore (b_1 a_2) = -(a_1 b_2).$$

(**b**) The interchange of two *subscripts*.

$(a_1 b_2) \equiv a_1 b_2 - a_2 b_1$. If the subscripts are interchanged, the second member becomes

$$a_2 b_1 - a_1 b_2 = -(a_1 b_2 - a_2 b_1) \therefore (a_2 b_1) = -(a_1 b_2).$$

2d. For determinants of the third order.

(**a**) The interchange of two *letters*.

$(a_1 b_2 c_3) \equiv (a_1 b_2) c_3 - (a_1 b_3) c_2 + (a_2 b_3) c_1$. In this, if we interchange a and b, the proposition is obvious from the first part of the demonstration, 1st, (**a**).

We have therefore to show that the proposition holds for b and c. We have, **5**,

$$(a_1 b_2)(a_2 c_3) - (a_2 b_3)(a_1 c_2) \equiv a_2 (a_1 b_2 c_3).$$

In this expression, interchanging b and c, the first member becomes $(a_1 c_2)(a_2 b_3) - (a_2 c_3)(a_1 b_2)$. Since a_2 remains unchanged, $(a_1 c_2 b_3) = -(a_1 b_2 c_3)$.

(**b**) The interchange of two *subscripts*.

$(a_1 b_2 c_3) \equiv (a_1 b_2) c_3 - (a_1 b_3) c_2 + (a_2 b_3) c_1$. (**L**). If the subscripts 2 and 3 are interchanged, the second member becomes $(a_1 b_3) c_2 - (a_1 b_2) c_3 + (a_3 b_2) c_1$. Since $(a_3 b_2) = -(a_2 b_3)$, 1st, (**b**), the second number of (**L**) becomes

$$-(a_1 b_2) c_3 + (a_1 b_3) c_2 - (a_2 b_3) c_1$$
$$\therefore (a_1 b_3 c_2) = -(a_1 b_2 c_3).$$

In the same manner it may be shown that the interchange of any other two subscripts in (**L**) changes the sign of the second member, $\therefore$ the proposition.

3d. For determinants of the fourth order.

(**a**) The interchange of two *letters*.

$$(a_1 b_2 c_3 d_4) \equiv (a_1 b_2 c_3) d_4 - (a_1 b_2 c_4) d_3 + (a_1 b_3 c_4) d_2 - (a_2 b_3 c_4) d_1 \quad (\mathbf{L}).$$

From 2d, (**a**), the proposition is obvious for an interchange of the first three letters. To show that the proposition holds for c and d, we have, **6**,

$$(a_1 b_2 c_3)\,(a_2 b_3 d_4) - (a_2 b_3 c_4)\,(a_1 b_2 d_3) = (a_2 b_3)\,(a_1 b_2 c_3 d_4).$$

The interchange of c and d transforms the minuend into subtrahend, and the subtrahend into minuend, in the first member. Hence, as $(a_2 b_3)$ remains unchanged, $(a_1 b_2 d_3 c_4) = -(a_1 b_2 c_3 d_4)$.

(**b**) The interchange of two *subscripts*.

$$(a_1 b_2 c_3 d_4) \equiv (a_1 b_2 c_3) d_4 - (a_1 b_2 c_4) d_3 + (a_1 b_3 c_4) d_2 - (a_2 b_3 c_4) d_1. \quad (\mathbf{M}).$$

In this, if we interchange the subscripts $_2$ and $_3$, the second member of (**M**) becomes

$$(a_1 b_3 c_2) d_4 - (a_1 b_3 c_4) d_2 + (a_1 b_2 c_4) d_3 + (a_3 b_2 c_4) d_1.$$

Now, by 2d, (**b**), $(a_1 b_3 c_2) = -(a_1 b_2 c_3)$; and $(a_3 b_2 c_4) = -(a_2 b_3 c_4)$. Hence the second member of (**M**) may be written

$$-(a_1 b_2 c_3) d_4 + (a_1 b_2 c_4) d_3 - (a_1 b_3 c_4) d_2 + (a_2 b_3 c_4) d_1,$$

and therefore $\qquad (a_1 b_3 c_2 d_4) = -(a_1 b_2 c_3 d_4)$.

In a similar manner the proposition may be established for the interchange of any other two subscripts.

It is obvious that two consecutive interchanges will leave the determinant unaltered either in sign or magnitude. Notice that an interchange of two letters corresponds to a uniform change in the order of succession of the unknown quantities in the original set of equations. Also, that an interchange of two subscripts corresponds to changing the order of the equations.

10. Applying the proposition of the preceding article to the values of x, y, z, and t, obtained in **8**, we have

$$x = \frac{(m_1 b_2 c_3 d_4)}{(a_1 b_2 c_3 d_4)}; \; y = \frac{(a_1 m_2 c_3 d_4)}{(a_1 b_2 c_3 d_4)}; \; z = \frac{(a_1 b_2 m_3 d_4)}{(a_1 b_2 c_3 d_4)}; \; t = \frac{(a_1 b_2 c_3 m_4)}{(a_1 b_2 c_3 d_4)}.$$

Notice that the common denominator in these values is the determinant of the fourth order, formed from the coefficients of the unknown quantities. Also, that the numerator of the value of x is obtained by changing the a of the denominator into m. The numerator of the value of y is likewise obtained by changing the b of the denominator into m, and that the numerators of the values of z and t are similarly obtained by changing the c and d into m respectively.

Notation.

11. We have seen that a determinant of the *second* order contains $2^2=4$ quantities, a determinant of the *third* order $3^2=9$ quantities, and a determinant of the *fourth* order $4^2=16$ quantities. It is customary to employ the notation introduced by Cayley, and write these determinants so that the quantities (called elements) entering into the determinant appear arranged in the form of a square, with a vertical line on each side.

Thus $(a_1 b_2) \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$; $(a_1 b_2 c_3) \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$; and $(a_1 b_2 c_3 d_4) \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

Other forms of notation are also $|a_1 b_2|$ for $(a_1 b_2)$; $|a_1 b_2 c_3|$ for $(a_1 b_2 c_3)$; $|a_1 b_2 c_3 d_4|$ for $(a_1 b_2 c_3 d_4)$.

There are still others to be described later. In Cayley's notation the elements are so arranged that, regarded as coefficients of the unknowns in the original set of equations, they occur in rows and columns in the regular order in which they are found in these original equations. Further, comparing the expansions with the square arrangement, we notice that each term contains *one*, and *only one*, element from each row and column, and that there is no other element from the same row and column in the same term. Hence, as already exemplified, there can be only 2, 3, or 4 elements in each term, according as the determinant is of the second, third, or fourth order. It will be noticed that the quantities occurring in the abbreviated

forms $(a_1 b_2 c_3)$, $(a_1 b_2)$, $|a_1 b_2 c_3 d_4|$, etc., are those found in one of the diagonals in the square arrangement, viz., the diagonal extending from the upper left-hand corner to the lower right-hand corner. This diagonal is called the *principal diagonal.* Similarly, that diagonal extending from the lower left-hand corner to the upper right-hand corner is the *secondary diagonal.* Any line parallel to these (principal or secondary) is a *minor diagonal.* Any of the expansions heretofore given show that the product of the elements of the principal diagonal is a positive term of the determinant. This term being composed of the elements of the principal diagonal, is called the *principal term.* The other terms can be formed from the principal term by making all the possible permutations of the subscripts and prefixing the proper sign to each permutation (**5** and footnote; also **6**).

Observe that the order of the letters in the abbreviated forms of notation is the order of the columns in the square arrangement, and that the order of the subscripts gives the order of the rows. Thus, $|a_1 b_2 c_3|$ means the determinant whose first column consists of a's, second column of b's, and third column of c's, and that the subscript of each letter in the first row is $_1$, and in the second each letter has the subscript $_2$, and in the third each letter has the subscript $_3$.

Illustrations are:

$$|a_3 b_2 c_4| \equiv \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

$$|a_3 b_4 c_5 d_1| \equiv \begin{vmatrix} a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \\ a_5 & b_5 & c_5 & d_5 \\ a_1 & b_1 & c_1 & d_1 \end{vmatrix}; \quad (a_1 c_2 b_3) \equiv \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}.$$

$$|a_r b_s c_u d_v| \equiv \begin{vmatrix} a_r & b_r & c_r & d_r \\ a_s & b_s & c_s & d_s \\ a_u & b_u & c_u & d_u \\ a_v & b_v & c_v & d_v \end{vmatrix}.$$

The expansion of determinants of the second and third orders.

12. Though we have already given the expansion of determinants of the second and third order several times, it will be useful here to compare these expansions with the square arrangement once more. Also, we are now prepared for a convenient mnemonic rule for the expansion of a determinant of the third order, to be given in **15**.

13. Since $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1 b_2 - a_2 b_1$, it is obvious that the expansion of a determinant of the second order is obtained by taking the product of the elements of the principal diagonal and the product of the elements in the secondary diagonal, and subtracting the second product from the first.

14. We have repeatedly shown that

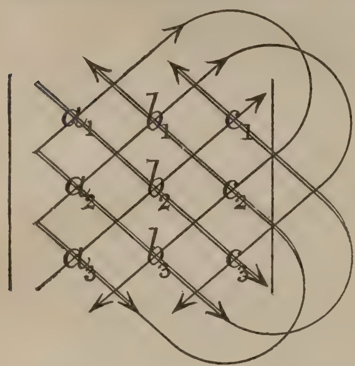
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} c_3 - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} c_2 + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} c_1.$$

From this it appears that a determinant of the third order can be decomposed into determinants of the second order, each multiplied by the elements in order of the last column, beginning with the last element. Since any column may be made the last, **9**, the assertion just made amounts to saying that a determinant of the third order may be expressed in terms of determinants of the second order and the elements of any column.

The reader will readily see how the determinant factors of the expansion in the present article are obtained from the original determinant. For example, the cofactor of c_2 is obtained by striking out the row and column in which c_2 is found, and regarding what is left as a determinant of the second order. Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

15. The following convenient rule for the complete expansion of a determinant of the third order is indicated in the accompanying diagram, and is described as follows: —



The terms composed of elements of the principal diagonal and of the minor diagonals parallel to it are positive, while those formed of elements in the secondary diagonal and the minor diagonals parallel to it are negative. The elements pierced by the double lines compose the positive terms. The elements pierced by the single lines similarly constitute the negative terms. In accordance with these directions, the expansion of

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ is } a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2.$$

This is identical with the expansion already obtained in **5**, as it should be.

EXAMPLES.

1. Find the values of:

$$\begin{vmatrix} -4 & -6 \\ 5 & 3 \end{vmatrix}; \quad \begin{vmatrix} 25 & 18 \\ 49 & 75 \end{vmatrix}; \quad \begin{vmatrix} a & b \\ b & a \end{vmatrix}; \quad \begin{vmatrix} a+b & b \\ a+b & a \end{vmatrix}; \quad \begin{vmatrix} a & -b & b \\ b & -c & c \end{vmatrix};$$

$$\begin{vmatrix} -10 & -6 \\ 8 & -3 \end{vmatrix}; \quad \begin{vmatrix} 7 & 1 \\ 5 & 0 \end{vmatrix}; \quad \begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix}; \quad \begin{vmatrix} 0 & 1-\frac{1}{a} \\ a & 1+\frac{1}{a} \end{vmatrix}.$$

2. Write in determinant form:

$7;\ 5;\ 16;\ -13;\ x_1y - xy;\ 3a - 7b;\ c^2 - bd;\ \frac{a}{b} - \frac{b}{a};\ 3gh - xy.$

(Suggestion: $-7 = 3 \times 2 - (1 \times -1) = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix}$. Numberless other forms could, of course, be given for the same quantity.)

3. Without passing from the determinant notation, show what relation exists between

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \text{ and } \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}. \quad \text{Also } \begin{vmatrix} x & y \\ m & n \end{vmatrix} \text{ and } \begin{vmatrix} m & n \\ x & y \end{vmatrix}. \quad \textbf{(9.)}$$

4. Compare $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$.

Also compare $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, $\begin{vmatrix} 3a & 3b \\ c & d \end{vmatrix}$, and $\begin{vmatrix} 3a & b \\ 3c & d \end{vmatrix}$.

5. Write the expansion of the following determinants:

$$(a_4 b_5);\ (a_1 b_k);\ |a_k b_l c_m|;\ |a_2 b_4 c_6|;\ (a_3 b_5 c_1);\ |b_2 c_3 a_1|.$$

6. Find the values of:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix};\ \begin{vmatrix} 4 & 7 & 8 \\ 0 & 3 & 6 \\ 0 & 5 & 9 \end{vmatrix};\ \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix};\ \begin{vmatrix} 0 & 0 & a \\ b & c & 0 \\ 0 & 0 & b \end{vmatrix};\ \begin{vmatrix} a & 0 & c \\ b & 0 & b \\ c & 0 & a \end{vmatrix};\ \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

7. Compare $\begin{vmatrix} a & 0 & c \\ d & 0 & f \\ g & 0 & k \end{vmatrix}$ and $\begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & k \end{vmatrix}$.

Also compare $\begin{vmatrix} a & mb & c \\ d & me & f \\ g & mh & k \end{vmatrix}$ and $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$.

Also compare $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

8. State the probable theorems exemplified by the results in Ex. 7.

9. Find the value of x in the equations:

(1) $\begin{vmatrix} x & 2 \\ 1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix}$; (2) $\begin{vmatrix} x & -4 & 1 \\ -6 & 3 & -2 \\ x & 2 & 1 \end{vmatrix} = 0$;

(3) $\begin{vmatrix} 1 & 1 & 1 \\ a & x & c \\ b & b & x \end{vmatrix} = 0$; (4) $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = - \begin{vmatrix} b & b & x \\ b & x & b \\ x & b & b \end{vmatrix}$.

10. Find the complete expansion of

$$\begin{vmatrix} a_r & b_r & c_r & d_r \\ a_t & b_t & c_t & d_t \\ a_n & b_n & c_n & d_n \\ a_o & b_o & c_o & d_o \end{vmatrix} \equiv |a_r b_t c_n d_o|. \quad (\mathbf{6}, \text{ equation } (\mathbf{R}), \textit{ et seq.})$$

11. Write in determinant form, square notation:

(1) $bfg + eid + hck - hfd - ecg - bik.$

(2) $m_1n_2r_3 - m_1n_3r_2 + m_2n_3r_1 - m_2n_1r_3 + m_3n_1r_2 - m_3n_2r_1.$

(3) $3xyz - x^3 - y^3 - z^3.$

12. Employ **9** to compare the following:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} \text{ and } \begin{vmatrix} d & e & f \\ g & h & k \\ a & b & c \end{vmatrix}; \quad \begin{vmatrix} m & n & o \\ p & q & r \\ s & t & u \end{vmatrix} \text{ and } \begin{vmatrix} o & n & m \\ r & q & p \\ u & t & s \end{vmatrix};$$

$$\begin{vmatrix} m & n & o \\ p & q & r \\ s & t & u \end{vmatrix} \text{ and } \begin{vmatrix} o & m & n \\ r & p & q \\ u & s & t \end{vmatrix}.$$

13. Expand the following in terms of determinants of the second order and the elements of any column (**14**). Verify the results by making use of the rule in **15**:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \quad \begin{vmatrix} x_2 & y_2 & m_2 \\ x_3 & y_3 & m_3 \\ x_4 & y_4 & m_4 \end{vmatrix}; \quad \begin{vmatrix} a_4 & b_4 & c_4 \\ a_3 & b_3 & c_3 \\ a_6 & b_6 & c_6 \end{vmatrix}.$$

14. Count the inversions of order in

(*a*) 1 3 5 4 2 6 7
(*b*) 2 3 6 1 4 5 7
(*c*) 6 3 5 4 1 7 2
(*d*) 7 8 9 6 5 3 4 2 1
(*e*) 9 8 7 6 5 4 3 2 1

CHAPTER II.

GENERAL PROPERTIES OF DETERMINANTS.

Notation and Definition.

16. The investigations of the preceding chapter have revealed the fact that a determinant of the second, third, or fourth order is a function of 2^2, 3^2, or 4^2 quantities respectively, and have also established a uniform law of formation for these functions. In order therefore to investigate the properties of Determinants in general, we have but to consider a function of n^2 quantities whose law of formation is given in the following definition.

17. DEFINITION. — A *Determinant* is always a function of n^2 quantities. These quantities, called elements, being arranged in the form of a square consisting of n rows, and thus also of n columns, n quantities in each row and in each column, the *determinant* of these n^2 quantities is the sum of the terms formed as follows:* Each term is the product of n elements, so chosen that there is one element from each row and one from each column, — but two elements from the same row or column must never occur in any one term. The sign-factor of each term is $(-1)^{p+q}$, in which p is the number of inversions of order† of the rows, and q is the number of inversions of order of the columns, from which the elements composing the term have been chosen.

Note. — Each term being composed of n factors, the determinant is said to be of the nth *order* or *degree*.

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* **22** *et seq.* will show that the law of formation given in this definition is the same as that already observed in determinants of the 2d, 3d, and 4th orders (**3** to **6** inclusive).

† **5**, footnote on *inversions of order*.

18. To expand $\begin{vmatrix} a & b & c \\ d & e & f \\ m & n & o \end{vmatrix}$ by the definition, we may select any row, as, for instance, the second row, and using each element* of that row in turn, according to the directions given, we shall form all the terms of the determinant. For the first term, then, taking d as the first element, we see that we can take b and o

$$\begin{vmatrix} a & b & c \\ d & e & f \\ m & n & o \end{vmatrix}$$

for the other factors of a term, and no more, since we have then chosen one element from each row and one from each column, and no two elements are from the same row or column. We now have the term dbo. To form another term containing d, we can evidently take n and c, giving the term dnc, which as before contains an element from each row and column, and no two elements are from the same row or column. No other terms containing d can be formed. The terms containing e are in the same way eao and mec; the diagram will sufficiently explain the manner of obtaining these terms.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ m & n & o \end{vmatrix}$$

The terms containing f are likewise naf and fbm.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ m & n & o \end{vmatrix}$$

To fix the signs of these terms, we will write under each term the numbers giving the rows and the numbers giving the

* There is a difference in the nomenclature. What we have called elements some authors call *constituents*, and an *element* is a term.

columns from which the elements have been taken, and opposite each series the number of inversions. Thus:

	dbo	*dnc*	*eao*	*mec*	*naf*	*fbm*
Rows	$213-1$	$231-2$	$213-1$	$321-3$	$312-2$	$213-1$
Columns	$123-0$	$123-0$	$213-1$	$123-0$	$213-1$	$321-3$

The sum of the inversions of order in rows and columns of the first term is unity; $\therefore (-1)^p = -1$, and dbo is negative. In dnc the sum of inversions of order in rows and columns is 2; $\therefore (-1)^p = 1$, and dnc is positive. Similarly for the other terms. Affecting the terms with their proper signs,

$$\begin{vmatrix} a & b & c \\ d & e & f \\ m & n & o \end{vmatrix} \equiv -dbo + dnc + eao - mec - naf + fbm.$$

Scholium. — This illustration is inserted only to give the reader a clear idea of the meaning of the definition, and not because we really employ the definition in the practical expansion of determinants. In fact, the great beauty of the determinant notation is that we are able to conduct most of our investigations with the help of determinants without requiring the expansions at all. In case it becomes necessary to expand a determinant, we have several excellent methods to be given later. One method for the expansion of a determinant of the third order has been given already (**15**).

19. In accordance with the notation already exemplified in Chapter I., a determinant of the nth order is written

$$\begin{vmatrix} a_1 & b_1 & c_1 & \dots & l_1 \\ a_2 & b_2 & c_2 & \dots & l_2 \\ a_3 & b_3 & c_3 & \dots & l_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & l_n \end{vmatrix}.$$

This form is shortened to $(a_1 b_2 c_3 \dots l_n)$ or $|a_1 b_2 c_3 \dots l_n|$, or to $\Sigma \pm a_1 b_2 c_3 \dots l_n$. In each of these shortened forms those elements occur which occupy the principal diagonal* in the square arrangement. The form $\Sigma \pm a_1 b_2 c_3 \dots l_n$ is suggestive of the manner in which the function is formed. The $\Sigma \pm$ stands for

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* **11**.

the sum of all the terms that can be formed from the principal term by permuting the subscripts and prefixing the proper sign to each. (**23**.)

Another and very convenient notation is obtained by employing a single letter affected with two subscripts; the first subscript giving the row, and the second subscript the column, in which the element occurs. Thus:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}.$$

This form may, like the first, be shortened to $|a_{11}\,a_{22}\dots a_{nn}|$, $(a_{11}\,a_{22}\,a_{33}\dots a_{nn})$, or $\Sigma \pm a_{11}\,a_{22}\,a_{33}\dots a_{nn}$. It may also be still further abbreviated to $|a_{1n}|$. A modification of this notation, with the two subscripts, consists in omitting the letter altogether, and writing the determinant thus:

$$\begin{vmatrix} (1,1) & (1,2) & (1,3) & \dots & (1,n) \\ (2,1) & (2,2) & (2,3) & \dots & (2,n) \\ (3,1) & (3,2) & (3,3) & \dots & (3,n) \\ \dots & \dots & \dots & \dots & \dots \\ (n,1) & (n,2) & (n,3) & \dots & (n,n) \end{vmatrix} \text{ or } \begin{vmatrix} 11 & 12 & 13 & \dots & 1n \\ 21 & 22 & 23 & \dots & 2n \\ 31 & 32 & 33 & \dots & 3n \\ \dots & \dots & \dots & \dots & \dots \\ n1 & n2 & n3 & \dots & nn \end{vmatrix};$$

or, finally, $\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$.

These last three forms are called the *umbral* notation.

20. The following corollaries flow from the definition in **17**. They are obvious upon a moment's reflection.

Cor. I. — The principal term is always positive.

Cor. II. — If each element of a row or of a column is zero, the determinant vanishes.

General Properties.

21. Theorem. — *If in a series of integers which are all different, any two are interchanged, the others remaining undisturbed, the number of inversions of order is thereby increased or diminished by an odd number.*

Let the series of integers be $Ae\ Bf\ C$, in which A is used to denote the series $\alpha\gamma\kappa \ldots$ preceding e, B denotes the series $hgl \ldots$ between e and f, and C the series following f.

In the first place, it is evident that if any two adjacent integers are interchanged, the number of inversions of order is thereby increased or diminished by unity. For let vm be any two adjacent integers in a series. If we write mv, we introduce one inversion of order if $m > v$. Or, if $m < v$, we have lost an inversion. Now, since this change cannot affect the rest of the series, we have increased or diminished the total number of inversions in the series by unity.

Again, in order to interchange e in $Ae\ Bf\ C$, with f separated from e by k, intervening elements, we may first interchange e with the elements to the right in regular succession $k+1$ times; this brings e into the place at first occupied by f. Then, in order to transfer f to the place formerly occupied by e, we have to pass f over k elements to the left. Altogether, we have changed the number of inversions of order from odd to even, or from even to odd, $2k+1$ (an odd number) of times. Hence the proposition.

22. THEOREM. — *The number of terms in a determinant of the nth order is* $1 \cdot 2 \cdot 3 \cdot \ldots n = n\,!$

The simplest way to form the terms of a determinant according to the definition, is to choose the elements from the columns in order; that is, the first element of a term from the first column, the second element from the second column, etc. Choosing the elements in this way, we may take the first element of a term from the *first* column and *third* row, say, the next element from the *second* column and *any* row except the *third*, the next element from the third column and any row except those already selected, and so on, until all the columns and rows have been drawn upon. The numbers of the rows from which the elements are chosen will constitute a permutation of the numbers 1, 2, 3, ... n, and it is obvious that we can therefore select the elements to form a term in as

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many different ways as there are permutations of the first n numbers, that is $n!$ There are accordingly $n!$ different terms.

23. Cor. I. — The terms of a determinant $|a_1 b_2 c_3 \dots l_n|$ may all be obtained by keeping the letters in alphabetical order (*i.e.*, choosing the elements for each term from the columns in order), making all the possible permutations of the subscripts, and prefixing the sign $+$ or $-$ to each permutation, according as the number of inversions of order is even or odd. Since the expansion of a determinant in accordance with the definition would also be obtained by keeping the rows in order, and choosing the elements from the columns in all possible ways, all the terms of $|a_1 b_2 c_3 \dots l_n|$ can be formed by permuting the letters, keeping the subscripts in order, and prefixing the sign $+$ or $-$ to each permutation, according as the number of inversions of the letters is even or odd.

24. Cor. II. — Similarly, the terms of $|a_{1n}|$ can be formed by making all the possible permutations of the first set of subscripts and keeping the second set in order; or the terms may be obtained by making all the possible permutations of the second set and leaving the first set in order.

Illustrations: To expand $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, we may write the permutations of the subscripts in a column, and indicate the number of inversions of order in each by a figure placed at the right; or we may write the permutations of the letters in the same way. Thus:

1 2 3 ... 0	$a\ b\ c$ ... 0
1 3 2 ... 1	$a\ c\ b$ ... 1
3 1 2 ... 2	$b\ a\ c$ ... 1
3 2 1 ... 3	$b\ c\ a$ ... 2
2 3 1 ... 2	$c\ a\ b$ ... 2
2 1 3 ... 1	$c\ b\ a$ ... 3

The two expansions are accordingly

$$a_1 b_2 c_3 - a_1 b_3 c_2 + a_3 b_1 c_2 - a_3 b_2 c_1 + a_2 b_3 c_1 - a_2 b_1 c_3,$$
$$a_1 b_2 c_3 - a_1 c_2 b_3 - b_1 a_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - c_1 b_2 a_3.$$

To expand $|a_{11}\,a_{22}\,a_{33}|$ according to Cor. II, we have simply to write the elements for each term with one set of subscripts in order; thus,

$$a_1\,a_2\,a_3,\ a_1\,a_2\,a_3,\ a_1\,a_2\,a_3,\ a_1\,a_2\,a_3,\ a_1\,a_2\,a_3,\ a_1\,a_2\,a_3\,;$$

and then for every term, according as we choose from columns or rows in order, write one permutation of the numbers 1, 2, 3, before or after the subscripts already written, obtaining

$$a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{32}\,a_{23} + a_{31}\,a_{12}\,a_{23} - a_{31}\,a_{22}\,a_{13} + a_{21}\,a_{32}\,a_{13} - a_{21}\,a_{12}\,a_{33}$$

or

$$a_{11}\,a_{22}\,a_{33} - a_{11}\,a_{23}\,a_{32} - a_{12}\,a_{21}\,a_{33} + a_{12}\,a_{23}\,a_{31} + a_{13}\,a_{21}\,a_{32} - a_{13}\,a_{22}\,a_{31}.$$

25. THEOREM. — *In any determinant, if the rows in order are made the columns in order, the determinant is unchanged.*

The theorem is an obvious consequence of **23** and **24**. The following proof is based directly upon the definition. Consider the determinants Δ and Δ', which differ only by making the rows of the one the columns of the other. Every term of Δ contains an element from each row and column of Δ; hence it contains an element from each column and row of Δ', and is therefore, disregarding the sign, also a term of Δ'. Similarly, every term of Δ' must be a term of Δ. We have now to show that the signs of corresponding terms are alike. Let the numbers of the rows and columns for a term of Δ be

$\alpha,\ \gamma,\ \beta,\ \tau,\ \sigma,\ \ldots$ for the rows;
$r,\ t,\ a,\ s,\ m,\ \ldots$ for the columns.

Then, by hypothesis, the numbers of the rows and columns of the corresponding term from Δ' will be

$r,\ t,\ a,\ s,\ m,\ \ldots$ for the rows;
$\alpha,\ \gamma,\ \beta,\ \tau,\ \sigma,\ \ldots$ for the columns.

The two terms obviously have the same sign. Hence the proposition.

?

Illustrations:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \equiv \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix};\quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}.$$

26. Theorem. — *In any determinant the number of positive terms equals the number of negative terms.*

By **23** all the terms of a determinant can be formed by keeping the letters in order, and making all the possible permutations of the subscripts (or **24**, case of the double subscripts, by keeping one set in order and permuting the other set). We have to show, therefore, that $\frac{n!}{2}$ permutations are even* and $\frac{n!}{2}$ are odd.* Let x and y be the number of even and odd permutations respectively; then $x+y=n!$ If we interchange any two subscripts in each of the x even permutations and in each of the y odd permutations, the even permutations become odd and the odd even. Since by the interchange of two subscripts we could only reproduce permutations all different from each other, and already found in the original set of permutations, it follows that $x=y$.

Not a Proof

27. Theorem. — *If two parallel lines (rows or columns) of a determinant are interchanged, the sign of the determinant is changed, but its numerical value is unchanged.*

Let Δ be the given determinant and Δ' the same determinant after the kth and rth rows have been interchanged. Then $-\Delta=\Delta'$.

Let $T \equiv \pm Ad_k Bm_r C$ be a term of Δ, in which A, B, and C denote the product of elements from all the rows and columns except the dth column and kth row, and the mth column and rth row. Then T (disregarding the sign) is also a term of Δ', for it contains an element from each row and column of Δ'. Now T, regarded as a term of Δ', contains exactly the same inversions of the columns as it does when regarded as a term of Δ; but the number of inversions in T, as to rows, when considered as a term of Δ', is an odd number, more or less, than when considered as a term of Δ. For, in writing the numbers of the rows, to determine the inversions, we write

* This language, of course, signifies permutations in which the number of inversions of order is even or odd respectively.

them just as we would for Δ, except that k and r will have changed places (d_k being found in the rth row, and m_r in the kth row of Δ'). Thus every term of Δ is found with the opposite sign in Δ', $\therefore -\Delta = \Delta'$. By **25** the proposition must be equally true for an interchange of two columns.

Illustrations:

$$-\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv -[a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_2b_1c_3 - a_1b_3c_2] \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = -\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \\ a_2 & b_2 & c_2 & d_2 \end{vmatrix} = -\begin{vmatrix} a_1 & d_1 & c_1 & b_1 \\ a_3 & d_3 & c_3 & b_3 \\ a_4 & d_4 & c_4 & b_4 \\ a_2 & d_2 & c_2 & b_2 \end{vmatrix}.$$

$$(a_1b_2c_3d_4) = -(a_2b_1c_3d_4) = (a_2b_3c_1d_4) = -(a_2b_3c_4d_1) = (a_3b_2c_4d_1).$$

28. Cor. — If two parallel lines of a determinant are identical, the determinant vanishes.

For, by the proposition, if the two identical rows or columns are interchanged, the sign of the determinant is changed. But the interchange of two identical lines cannot affect the determinant. Therefore

$$\Delta = -\Delta,$$
$$2\,\Delta = 0, \text{ or } \Delta = 0.$$

Illustrations:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ a & b & c \end{vmatrix} = aec + dbc + abf - aec - dbc - abf = 0.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_2 & b_2 & c_2 & d_2 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_2 & a_4 \\ b_1 & b_2 & b_2 & b_4 \\ c_1 & c_2 & c_2 & c_4 \\ d_1 & d_2 & d_2 & d_4 \end{vmatrix} = 0.$$

$$(a_1\,b_2\,c_2\,d_4) = 0. \qquad |\,a_1\,b_2\,a_3\,d_4\,| = 0.$$

29. If in a series of integers,

$$f,\ a,\ d,\ c,\ l,\ m,\ n,$$

the first is passed over all the others in succession to become the last, the others remaining undisturbed, thus,

$$a,\ d,\ c,\ l,\ m,\ n,\ f,$$

the numbers are said to have been *cyclically* interchanged. It is obvious that a cyclical permutation of n given numbers can always be effected by $n-1$ interchanges of two adjacent numbers. Accordingly, a permutation containing an odd or even number of inversions still contains, after a cyclical interchange, an odd or even number of inversions if n is odd; if n is even, however, a permutation containing an odd or even number of inversions will, after a cyclical interchange, contain an even or odd number of inversions respectively.

From a given permutation of n integers any other permutation can be obtained by cyclical interchanges. Thus, from

$$f\ a\ d\ c\ e\ g\ b$$

we get

$$c\ a\ g\ f\ d\ b\ e$$

as follows: —

$$\begin{array}{c} c\ f\ a\ d\ e\ g\ b \\ c\ a\ f\ d\ e\ g\ b \\ c\ a\ g\ f\ d\ e\ b \\ c\ a\ g\ f\ d\ b\ e \end{array}$$

The groups in which the cyclical interchanges take place are, of course, *fadc*, *fa*, *fdeg*, *f*, *d*, *eb*.

30. The previous article (or **27**) establishes the following theorem:

Theorem. — If in a determinant Δ any row or column be passed over k rows or columns in succession, and the resulting determinant be denoted by Δ', then

$$\Delta = (-1)^k \Delta'.$$

Illustrations:

$$\begin{vmatrix} x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \\ x_3 & y_3 & z_3 & t_3 \\ x_4 & y_4 & z_4 & t_4 \end{vmatrix} = \begin{vmatrix} x_3 & y_3 & z_3 & t_3 \\ x_1 & y_1 & z_1 & t_1 \\ x_2 & y_2 & z_2 & t_2 \\ x_4 & y_4 & z_4 & t_4 \end{vmatrix} = \begin{vmatrix} x_3 & t_3 & y_3 & z_3 \\ x_1 & t_1 & y_1 & z_1 \\ x_2 & t_2 & y_2 & z_2 \\ x_4 & t_4 & y_4 & z_4 \end{vmatrix} = - \begin{vmatrix} x_1 & t_1 & y_1 & z_1 \\ x_2 & t_2 & y_2 & z_2 \\ x_4 & t_4 & y_4 & z_4 \\ x_3 & t_3 & y_3 & z_3 \end{vmatrix}.$$

$$|\,x_0\ y_1\ v_2\ w_3\,| = -\,|\,x_1\ y_2\ v_3\ w_0\,| = -\,|\,v_1\ x_2\ y_3\ w_0\,| = |\,x_1\ y_2\ w_3\ v_0\,|.$$

EXAMPLES.

1. The student who has not done the examples at the end of the first chapter may attend to them before proceeding to the following.

2. What terms of $|a_1 b_2 c_3 d_4|$ contain $b_2 d_3$?

3. Write the terms of $(x_1 y_2 w_3 z_4 t_5)$ that contain $t_1 y_4 w_3$.

4. Show that in a determinant of the nth order only two terms can have $(n-2)$ elements in common, and that these terms have opposite signs.

5. What is the sign-factor of the term containing the elements in the secondary diagonal of a determinant of the nth order?

6. Show that the sign of a term is independent of the arrangement of the elements composing it.

7. Show that the sign of a determinant is not changed by any interchanges of rows and columns that leave the same elements in the principal diagonal, whatever the final arrangement of the elements in this diagonal.

Sug. If $a_{\beta\beta}\, a_{\gamma\gamma}\, a_{\alpha\alpha} \ldots a_{\lambda\lambda}$ be the final arrangement sought, $a_{\beta\beta}$ can be brought into the first place by $2(\beta-1)$ interchanges of two rows and columns, etc.

8. A corollary from **30** is: Any element a_{ik} can be transferred to the first place by making the ith row and kth column the first row and column, and then multiplying the determinant by $(-1)^{i+k}$.

31. Theorem. — *If every element of any line (row or column) is multiplied by any number, the determinant is multiplied by that number.*

Since every term of the determinant contains one element, and only one, from the line mentioned in the theorem, the truth of the proposition is evident ?

Illustrations:

$$\begin{vmatrix} a_1r & b_1r & c_1r \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = r\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1r & c_1 \\ a_2 & b_2r & c_2 \\ a_3 & b_3r & c_3 \end{vmatrix} = r^2\begin{vmatrix} \frac{a_1}{r} & b_1 & c_1 \\ \frac{a_2}{r} & b_2 & c_2 \\ \frac{a_3}{r} & b_3 & c_3 \end{vmatrix}.$$

$$\Delta \equiv \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}; \text{ then } abc\Delta = \begin{vmatrix} abc & a^2 & a^3 \\ abc & b^2 & b^3 \\ abc & c^2 & c^3 \end{vmatrix}; \therefore \Delta = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}.$$

Let the student show that

$$\begin{vmatrix} bcd & a & a^2 & a^3 \\ cda & b & b^2 & b^3 \\ dab & c & c^2 & c^3 \\ abc & d & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 & a^4 \\ 1 & b^2 & b^3 & b^4 \\ 1 & c^2 & c^3 & c^4 \\ 1 & d^2 & d^3 & d^4 \end{vmatrix}.$$

32. Cor. I. — Changing the signs of all the elements of any row or column changes the sign of the determinant; for it is equivalent to multiplying the determinant by -1.

33. Cor. II. — If two rows or two columns differ only by a constant factor, the determinant vanishes. For we may divide each element by the constant factor, and write this factor as a multiplier before the determinant. Then the determinant vanishes by **28**.

Illustrations:

$$\begin{vmatrix} a & 1 & a^n \\ b & a & a^{n+1} \\ c & a^2 & a^{n+2} \end{vmatrix} = a^n\begin{vmatrix} a & 1 & 1 \\ b & a & a \\ c & a^2 & a^2 \end{vmatrix} = 0. \qquad \begin{vmatrix} 1 & 5 & 7 \\ 2 & 10 & 6 \\ 3 & 15 & 9 \end{vmatrix} = 5\begin{vmatrix} 1 & 1 & 7 \\ 2 & 2 & 6 \\ 3 & 3 & 9 \end{vmatrix} = 0.$$

34. Theorem. — *If each element of any line* of a determinant is a binomial, the determinant equals the sum of two determinants; the first of which is obtained from the given determinant by substituting for the binomial elements the first terms of the binomials, and the second determinant is obtained from the given determi-*

* Since it has been shown (**25**) that what is true of the rows of a determinant holds for the columns, it will only be necessary hereafter to state a proposition with reference to *either* rows or columns.

nant by substituting for the binomial elements the second terms of the binomials.

By the definition every term of the determinant must contain one of the binomial elements.

Let $$(m+n)\ b\ g\ h\ k \ldots l$$
be one of the terms of the given determinant; this may be written $$m\ b\ g\ h\ k \ldots l + n\ b\ g\ h\ k \ldots l.$$
Now the first term of this sum is a term of the original determinant, with m written for $m+n$, and the second term is a term of the original determinant, with n written for $m+n$. It is obvious that a similar statement applies to every term of the given determinant; hence the proposition.

Illustrations:

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}.$$

$$\begin{vmatrix} x_1-y_1 & m_1 & n_1 \\ x_2-y_2 & m_2 & n_2 \\ x_3-y_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} x_1 & m_1 & n_1 \\ x_2 & m_2 & n_2 \\ x_3 & m_3 & n_3 \end{vmatrix} - \begin{vmatrix} y_1 & m_1 & n_1 \\ y_2 & m_2 & n_2 \\ y_3 & m_3 & n_3 \end{vmatrix}.$$

35. The preceding theorem is evidently capable of extension. The same reasoning applies to a determinant any line of which is composed of *polynomial* elements, or, again, in which each element of every line is a polynomial. That is to say: *If each element of any row is a polynomial of q terms, each element of another row a polynomial of r terms, each element of another row a polynomial of s terms, etc., the given determinant is the sum of* $s \times q \times r \ldots$ *determinants.* Thus:

$$\begin{vmatrix} a+b+c & m-n & t \\ d+e-f & o+p & u \\ g-h+k & q-r & v \end{vmatrix} = \begin{vmatrix} a & m-n & t \\ d & o+p & u \\ g & q-r & v \end{vmatrix} + \begin{vmatrix} b & m-n & t \\ e & o+p & u \\ -h & q-r & v \end{vmatrix}$$

$$+ \begin{vmatrix} c & m-n & t \\ -f & o+p & u \\ k & q-r & v \end{vmatrix} = \begin{vmatrix} a & m & t \\ d & o & u \\ g & q & v \end{vmatrix} + \begin{vmatrix} a & -n & t \\ d & p & u \\ g & -r & v \end{vmatrix} + \begin{vmatrix} b & m & t \\ e & o & u \\ -h & q & v \end{vmatrix}$$

$$+ \begin{vmatrix} b & -n & t \\ e & p & u \\ -h & -r & v \end{vmatrix} + \begin{vmatrix} c & m & t \\ -f & o & u \\ k & q & v \end{vmatrix} + \begin{vmatrix} c & -n & t \\ -f & p & u \\ k & -r & v \end{vmatrix}.$$

36. Reciprocally, *If q determinants differ from each other only in a single line, the sum of these determinants is a single determinant, derived from any of the given determinants by substituting for the elements of the line which is different in each of the q determinants, the sum of the corresponding elements of the q determinants.*

Illustrations:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix} + \begin{vmatrix} a & b & c \\ x & y & z \\ g & h & k \end{vmatrix} + \begin{vmatrix} m & n & o \\ a & b & c \\ g & h & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ d+x-m & e+y-n & f+z-o \\ g & h & k \end{vmatrix}.$$

The student may show that

$$\begin{vmatrix} a_1 & a_1 & b_1 & c_1 \\ a_2 & a_2 & b_2 & c_2 \\ 0 & a_3 & b_3 & c_3 \\ 0 & a_4 & b_4 & c_4 \end{vmatrix} - \begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & a_3 \\ a_4 & b_4 & c_4 & a_4 \end{vmatrix} = 0.$$

37. Theorem. — *A determinant remains unchanged if the elements of any line be increased or diminished by equal multiples of the corresponding elements of any parallel line.*

We are to show that

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & \dots & l_1 \\ a_2 & b_2 & c_2 & d_2 & \dots & l_2 \\ a_3 & b_3 & c_3 & d_3 & \dots & l_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & d_n & \dots & l_n \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \pm q_1 a_1 \pm q_2 b_1 \pm \dots \pm l_1 & d_1 & \dots & l_1 \\ a_2 & b_2 & c_2 \pm q_1 a_2 \pm q_2 b_2 \pm \dots \pm l_2 & d_2 & \dots & l_2 \\ a_3 & b_3 & c_3 \pm q_1 a_3 \pm q_2 b_3 \pm \dots \pm l_3 & d_3 & \dots & l_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n \pm q_1 a_n \pm q_2 b_n \pm \dots \pm l_n & d_n & \dots & l_n \end{vmatrix}.$$

Calling the first determinant Δ, and the second Δ', we have, **35**,

$$\Delta' = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & \dots & l_1 \\ a_2 & b_2 & c_2 & d_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & d_n & \dots & l_n \end{vmatrix} \pm q_1 \begin{vmatrix} a_1 & b_1 & a_1 & d_1 & \dots & l_1 \\ a_2 & b_2 & a_2 & d_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & a_n & d_n & \dots & l_n \end{vmatrix}$$

$$\pm q_2 \begin{vmatrix} a_1 & b_1 & b_1 & d_1 & \dots & l_1 \\ a_2 & b_2 & b_2 & d_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & b_n & d_n & \dots & l_n \end{vmatrix} \pm \quad \dots\dots\dots\dots\dots\dots \quad \pm \begin{vmatrix} a_1 & b_1 & l_1 & d_1 & \dots & l_1 \\ a_2 & b_2 & l_2 & d_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & l_n & d_n & \dots & l_n \end{vmatrix}.$$

Whence, since all the determinants of this series, except the first, vanish, $\Delta = \Delta'$.

This theorem is of great importance in simplifying and expanding determinants. Thus:

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0.$$

$$\begin{vmatrix} a & a+3 & a+6 \\ a+1 & a+4 & a+7 \\ a+2 & a+5 & a+8 \end{vmatrix} = \begin{vmatrix} 3(a+1) & 3(a+4) & 3(a+7) \\ a+1 & a+4 & a+7 \\ a+2 & a+5 & a+8 \end{vmatrix} = 0.$$

The second determinant is obtained by adding the second and third rows to the first row.

$$\begin{vmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{vmatrix} = - \begin{vmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{vmatrix} = -8 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -16.$$

The second determinant is obtained by adding the first row to each of the others.

The third determinant is obtained from the second by observing that as all the elements, except one, of the first column of that determinant are zeros, all the terms vanish that do not contain (-1).

$$\begin{vmatrix} 7 & 11 & 4 \\ 13 & 15 & 10 \\ 3 & 9 & 6 \end{vmatrix} = 3 \begin{vmatrix} 7 & 11 & 4 \\ 13 & 15 & 10 \\ 1 & 3 & 2 \end{vmatrix} = 3 \begin{vmatrix} 7 & -10 & -10 \\ 13 & -24 & -16 \\ 1 & 0 & 0 \end{vmatrix} = 3 \begin{vmatrix} 10 & 10 \\ 24 & 16 \end{vmatrix}$$

$$= 30\,(16-24) = -240.$$

The third determinant is obtained from the second by subtracting three times the first column from the second column, and twice the first column from the third.

Minor Determinants.

38. If in a determinant any number of rows and the same number of columns are suppressed, the determinant consisting of the remaining elements (their relative positions being undisturbed) is called a *minor* of the given determinant.

If one row and one column are suppressed, the result is a *principal* minor, or a *first* minor; if two rows and two columns

have been suppressed, a *second* minor; and so on. The elements common to the suppressed rows and columns also form a determinant called the *complementary* of the minor, formed from the rows and columns that were left undisturbed in the original determinant.

Thus, $\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$ and $\begin{vmatrix} c_2 & d_2 \\ c_4 & d_4 \end{vmatrix}$ are complementary minors of $|a_1\, b_2\, c_3\, d_4|$; also $|a_1\, b_2|$ and $|c_3\, d_4|$, or b_2 and $|a_1\, c_3\, d_4|$, are complementary minors of $|a_1\, b_2\, c_3\, d_4|$. $|c_1\, d_3|$ and $|a_2\, b_4\, e_5|$ are complementary minors of $|a_1\, b_2\, c_3\, d_4\, e_5|$. In general, if the determinant is of the nth order, two complementary minors will be of the rth and $(n-r)$th orders respectively. A determinant of the nth order has n^2 first minors, $\dfrac{n^2(n-1)^2}{(2\,!)^2}$ second minors, etc.

Since we usually denote a determinant by Δ, it is convenient to denote the minor obtained by suppressing the row and column of a_3 by Δ_{a_3}; that obtained by suppressing the row and column of d_6 by Δ_{d_6}, etc.

Similarly, a second minor, obtained by suppressing the rows and columns of b_k and c_r, is denoted by Δ_{b_k, c_r}; and so on.

Equally efficient notations are: $D\begin{cases} l \\ m, \end{cases}$ and $D\begin{cases} l \; n \; t \\ m \; r \; k, \end{cases}$ for the minor obtained by suppressing the lth row and mth column, and the minor obtained by suppressing the lth, nth, and tth rows, the mth, rth, and kth columns respectively of $|a_{1n}|$.

39. Since, by definition, every term of a determinant contains one, and only one element from any line, the determinant must be a linear homogenous function of the elements of any one row or column. Thus:

$$\begin{aligned} |a_1\, b_2\, c_3 \dots l_n| &= a_1 A_1 + a_2 A_2 + a_3 A_3 + \cdots + a_n A_n \\ &= a_1 A_1 + b_1 B_1 + c_1 C_1 + \cdots + l_1 L_1 \\ &= c_1 C_1 + c_2 C_2 + c_3 C_3 + \cdots + c_n C_n \\ &= \dots \quad \dots \quad \dots \quad \dots \quad \dots, \end{aligned}$$

in which $A_1, A_2 \dots A_n$; $C_1, C_2 \dots C_n$; etc., denote functions of the elements found in the rows and columns outside of the particu-

lar line, in terms of which the development is given. In the next article we shall find the values of these functions. Since we may regard the determinant as a function of n^2 independent quantities, each of the coefficients A_1, A_2, ..., may be obtained by differentiating $|a_1 b_2 c_3 \ldots l_n|$ with reference to the quantity whose coefficient is desired. Introducing this concept, the equations above written become respectively

$$\begin{aligned} |a_1 b_2 c_3 \ldots l_n| &= a_1 \frac{d\Delta}{da_1} + a_2 \frac{d\Delta}{da_2} + a_3 \frac{d\Delta}{da_3} + \cdots + a_n \frac{d\Delta}{da_n} \\ &= a_1 \frac{d\Delta}{da_1} + b_1 \frac{d\Delta}{db_1} + c_1 \frac{d\Delta}{dc_1} + \cdots + l_1 \frac{d\Delta}{dl_1} \\ &= c_1 \frac{d\Delta}{dc_1} + c_2 \frac{d\Delta}{dc_2} + c_3 \frac{d\Delta}{dc_3} + \cdots + c_n \frac{d\Delta}{dc_n} \\ &= \quad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots \end{aligned}$$

This notation is often employed.

40. THEOREM. — *The coefficient of any element in the expansion of a determinant is the first minor obtained by suppressing the row and column to which the element belongs. This minor is taken with the + sign, if the sum of the row and column numbers, to which the element belongs, is even; with the − sign, if this sum is odd.*

Consider the determinant $\Delta \equiv \Sigma \pm a_1 b_2 c_3 \ldots l_n$, and suppose Δ to be written,

$$\Delta = a_1 A_1 + a_2 A_2 + a_3 A_3 + \cdots + a_n A_n. \tag{1}$$

We can collect all the terms of Δ that contain a_1, and write this element as a factor of the polynomial that results; we can do the same for a_2, a_3, and so on, for each element of the first column. These polynomials are A_1, A_2, A_3, etc. Now since A_1 is the cofactor of a_1, it can contain no elements from the first row or column; hence $a_1 A_1$ can be obtained from $\Sigma \pm a_1 b_2 c_3 \ldots l_n$ by considering a_1 as fixed, and making all the possible permutations of the subscripts of the remaining letters, *i.e.*, by multiplying a_1 by $\Sigma \pm b_2 c_3 \ldots l_n$.

Hence, $A_1 = \Delta_{a_1}$, the minor obtained by suppressing the first row and first column of Δ.

Now, we can bring a_2 into the first place by one interchange of rows: we then have $\Delta = -\Sigma \pm a_2 b_1 c_3 \ldots l_n$. Employing the same reasoning as before, A_2 must be obtained by multiplying a_2 by $-\Sigma \pm b_1 c_3 \ldots l_n$; whence $A_2 = -\Delta_{a_2}$. Again, a_3 can be brought into the first place by two interchanges of two rows; whence $A_3 = \Delta_{a_3}$, and so on. Finally, a_n can be brought into the first place by $n-1$ interchanges of two rows; hence, as before, $A_n = (-1)^{n-1}\Delta_{a_n}$.

Substituting these values of A_1, A_2, etc., in (1),

$$\Delta = a_1\Delta_{a_1} - a_2\Delta_{a_2} + a_3\Delta_{a_3} - \cdots + (-1)^{n-1}a_n\Delta_{a_n}.$$

Since the columns may be made rows, it is evident that

$$\Delta = a_1\Delta_{a_1} - b_1\Delta_{b_1} + c_1\Delta_{c_1} - \cdots + (-1)^{n-1}l_1\Delta_{l_1}.$$

It remains to be shown that the proposition holds for an element not in the first row or column, that is, $A_{ik} = (-1)^{i+k}\Delta_{a_{ik}}$, for the coefficient of the element in the ith row and kth column. We may transfer the ith row to the first place by $i-1$ interchanges of two rows, and the kth column may likewise be made the first by $k-1$ interchanges of two columns. The element under consideration is now in the first place. Calling the transformed determinant Δ', we have

$$\Delta = (-1)^{i+k-2}\Delta', \text{ or } \Delta = (-1)^{i+k}\Delta'.$$

Whence $\quad A_{ik} = (-1)^{i+k}\Delta_{a_{ik}}.$

41. Cor. I. — A determinant can be developed in terms of the elements of any line and their principal minors. The signs are alternately + and −; and the first term is + or −, according as the number of the line is odd or even.

Illustrations:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

$$= -b_1|a_2\ c_3\ d_4| + b_2|a_1\ c_3\ d_4| - b_3|a_1\ c_2\ d_4| + b_4|a_1\ c_2\ d_3|$$

$$= \quad c_1|a_2\ b_3\ d_4| - c_2|a_1\ b_3\ d_4| + c_3|a_1\ b_2\ d_4| - c_4|a_1\ b_2\ d_3|$$

$$= -a_4|b_1\ c_2\ d_3| + b_4|a_1\ c_2\ d_3| - c_4|a_1\ b_2\ d_3| + d_4|a_1\ b_2\ c_3|$$

$$= \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

42. **41** obviously gives a ready way of expanding any determinant.* For we may express the given determinant in terms of the elements of any line and their principal minors; these minors will be determinants of the $(n-1)$th order. By a second application of **41**, each of the minors in the first expansion may be expressed in terms of the elements of any line and their principal minors, which minors will be of the $(n-2)$th order. So by successive application of **41**, any determinant may be expressed in terms of determinants of the second order; and these latter, being binomials, can be at once written out. Thus:

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 1\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$$

$$= 1(15-16) - 2(10-12) + 3(8-9) = 0.$$

$$|a_1 b_2 c_3 d_4|^{\dagger} = a_1|b_2 c_3 d_4| - a_2|b_1 c_3 d_4| + a_3|b_1 c_2 d_4| - a_4|b_1 c_2 d_3|$$

$$= a_1[b_2|c_3 d_4| - b_3|c_2 d_4| + b_4|c_2 d_3|] - a_2[b_1|c_3 d_4| - b_3|c_1 d_4| + b_4|c_1 d_3|]$$

$$+ a_3[b_1|c_2 d_4| - b_2|c_1 d_4| + b_4|c_1 d_2|] - a_4[b_1|c_2 d_3| - b_2|c_1 d_3| + b_3|c_1 d_2|]$$

$$= a_1 b_2 c_3 d_4 - a_1 b_2 c_4 d_3 - a_1 b_3 c_2 d_4 + a_1 b_3 c_4 d_2 + a_1 b_4 c_2 d_3 - a_1 b_4 c_3 d_2$$

$$- a_2 b_1 c_3 d_4 + a_2 b_1 c_4 d_3 + \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$+ a_3 b_1 c_2 d_4 - a_3 b_1 c_4 d_2 - \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$- a_4 b_1 c_2 d_3 + a_4 b_1 c_3 d_2 + a_4 b_2 c_1 d_3 - a_4 b_2 c_3 d_1 - a_4 b_3 c_1 d_2 + a_4 b_3 c_2 d_1$$

43. As another corollary from **40**, it is evident that if all the elements of any row of a determinant except one are zeros, the determinant equals this element into its corresponding minor, taken with the proper sign. Thus, if the element is in the ith row and kth column, *i.e.*, a_{ik}, then $\Delta = (-1)^{i+k} a_{ik} \Delta_{a_{ik}}$.

Illustrations:

$$\begin{vmatrix} 5 & 6 & 4 & 3 \\ 2 & 0 & 0 & 0 \\ 3 & 1 & 1 & 1 \\ 4 & 1 & 2 & 1 \end{vmatrix} = -2\begin{vmatrix} 6 & 4 & 3 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = -2\begin{vmatrix} 0 & -2 & -3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} -2 & -3 \\ 1 & 0 \end{vmatrix} = 6.$$

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* Compare **15**. † Compare **6**.

The student may establish the following:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2 & c_2 & d_2 \\ 0 & 0 & c_3 & d_3 \\ 0 & 0 & 0 & d_4 \end{vmatrix} = a_1 b_2 c_3 d_4. \qquad \begin{vmatrix} 0 & 0 & 0 & d_1 \\ 0 & 0 & c_2 & d_2 \\ 0 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_4 b_3 c_2 d_1.$$

44. From the last two examples it appears *that if all the elements on one side of either diagonal are zeros, the determinant reduces to a single term, viz., the term composed of the elements in the diagonal which contains no zero elements.*

EXAMPLES.

1. Show that the following determinant vanishes: $\begin{vmatrix} 3 & 1 & 5 & 2 \\ 2 & 5 & 7 & 3 \\ 8 & 9 & 1 & 4 \\ 6 & 15 & 21 & 9 \end{vmatrix}$.

2. $\begin{vmatrix} 2 & 1 & -7 \\ -4 & -3 & 8 \\ 6 & 5 & -9 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 7 \\ 2 & 3 & 8 \\ 3 & 5 & 9 \end{vmatrix}$.

3. Show that $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = \dfrac{1}{\alpha\beta\gamma} \begin{vmatrix} 1 & 1 & 1 \\ \alpha'\beta\gamma & \beta'\gamma\alpha & \gamma'\alpha\beta \\ \alpha''\beta\gamma & \beta''\gamma\alpha & \gamma''\alpha\beta \end{vmatrix}$.

This can be readily established by multiplying the columns by $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, respectively, and then dividing the first row by $\alpha\beta\gamma$. *A similar reduction can be effected, in general, whenever it is desired to reduce a determinant to one in which the elements of one line are units.*

4. Find the expansion of $\Delta \equiv \begin{vmatrix} 4 & 2 & 5 & 10 \\ 1 & 1 & 6 & 3 \\ 7 & 3 & 0 & 5 \\ 0 & 2 & 5 & 8 \end{vmatrix}$.

We notice that 20 is the L.C.M. of the elements in the first row; hence, multiplying the columns in order by 5, 10, 4, 2, there results

$$\Delta = \frac{1}{5 \cdot 10 \cdot 4 \cdot 2} \begin{vmatrix} 20 & 20 & 20 & 20 \\ 5 & 10 & 24 & 6 \\ 35 & 30 & 0 & 10 \\ 0 & 20 & 20 & 16 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 5 & 10 & 24 & 6 \\ 7 & 6 & 0 & 2 \\ 0 & 5 & 5 & 4 \end{vmatrix}.$$

Now, subtracting four times the first row from the fourth row, two times the first row from the third row, and six times the first row from the second row, the last determinant becomes

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & 18 & 0 \\ 5 & 4 & -2 & 0 \\ -4 & 1 & 1 & 0 \end{vmatrix} = - \begin{vmatrix} -1 & 4 & 18 \\ 5 & 4 & -2 \\ -4 & 1 & 1 \end{vmatrix} = - \begin{vmatrix} 71 & -14 & 18 \\ -3 & 6 & -2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= -6 \begin{vmatrix} 71 & -7 \\ -1 & 1 \end{vmatrix} = -6 \begin{vmatrix} 64 & -7 \\ 0 & 1 \end{vmatrix} = -384.$$

*5. $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \dfrac{1}{a_1 a_2 a_3} \begin{vmatrix} 1 & 1 & 1 \\ a_2 a_3 b_1 & a_1 a_3 b_2 & a_1 a_2 b_3 \\ a_2 a_3 c_1 & a_1 a_3 c_2 & a_1 a_2 c_3 \end{vmatrix}$;

also $\begin{vmatrix} 0 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & a_3 b_2 c_1 & a_2 b_3 c_1 \\ 1 & a_3 b_1 c_2 & a_2 b_1 c_3 \end{vmatrix}$.

6. $\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \\ 1 & \gamma & \gamma^2 \end{vmatrix} = (\beta - \gamma)\ (\gamma - \alpha)\ (\alpha - \beta)$.

Δ vanishes if $\alpha = \beta$, or $\beta = \gamma$, or $\alpha = \gamma$; hence $\alpha - \beta$, $\beta - \gamma$, and $\alpha - \gamma$ must be factors of Δ. Now the product of the three differences is a function of the third degree in α, β, γ; so is Δ; hence the product of the three differences can differ from Δ only by a constant factor. Comparing the term $\beta\gamma^2$ (the principal term), we see the factor mentioned is $+1$.

7. Show that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix} = -(\beta - \gamma)(\alpha - \delta)(\gamma - \alpha)(\beta - \delta)(\alpha - \beta)(\gamma - \delta).$$

Notice that Examples 6 and 7 give in determinant form the product of the differences of the roots of an equation whose roots are α, β, γ, ...

8. Expand $\begin{vmatrix} 8 & 7 & 2 & 20 \\ 3 & 1 & 4 & 7 \\ 5 & 0 & 11 & 0 \\ 8 & 1 & 0 & 6 \end{vmatrix}$; also $\begin{vmatrix} 1 & \alpha & -\beta \\ -\alpha & 1 & \gamma \\ \beta & -\gamma & 1 \end{vmatrix}$.

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* Compare example 3.

Expand the first determinant in terms of the elements of the third row and their principal minors, since two of these elements are zero; then observe that two elements in a row of each of the resulting determinants are unity; hence, each determinant can be readily reduced to one of the next lower order by **35** and **42**.

9. $\begin{vmatrix} 0 & c & b & d \\ c & 0 & a & e \\ b & a & 0 & f \\ d & e & f & 0 \end{vmatrix} = a^2d^2 + b^2e^2 + c^2f^2 - 2bcef - 2cafd - 2abde.$

10. Expand $\begin{vmatrix} -a & b & c & d \\ b & -a & d & c \\ c & d & -a & b \\ d & c & b & -a \end{vmatrix}$; also $\begin{vmatrix} 1 & \alpha & \beta & \gamma \\ -\alpha & 1 & \gamma' & -\beta' \\ -\beta & -\gamma' & 1 & \alpha' \\ -\gamma & \beta & -\alpha' & 1 \end{vmatrix}$.

11. Establish the following identity, and express either determinant as the product of four linear factors:

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & z^2 & y^2 \\ 1 & z^2 & 0 & x^2 \\ 1 & y^2 & x^2 & 0 \end{vmatrix} = \begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}.$$

12. Simplify $\begin{vmatrix} a_1+h_1+k_1 & a_2+h_1+k_2 & a_3+h_1+k_3 & 1 \\ b_1+h_2+k_1 & b_2+h_2+k_2 & b_3+h_2+k_3 & 1 \\ c_1+h_3+k_1 & c_2+h_3+k_2 & c_3+h_3+k_3 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$.

13. Show that $\begin{vmatrix} x & y+z+t & x+y & z+t \\ y & z+t+x & y+z & t+x \\ z & t+x+y & z+t & x+y \\ t & x+y+z & t+x & y+z \end{vmatrix} = 0.$

14. Express as a single determinant:

(1) $|a_1\, b_4\, c_5| + |a_2\, b_4\, c_5| - |a_3\, b_4\, c_5|$.

(2) $|a_0\, b_2\, c_5| - |a_0\, b_3\, c_5| - |a_1\, b_3\, c_5| + |a_1\, b_2\, c_5|$.

15. $\begin{vmatrix} a_1+a_2+a_3 & a_2+a_3+a_4 & a_3+a_4+a_1 & a_4+a_1+a_2 \\ b_1+b_2+b_3 & b_2+b_3+b_4 & b_3+b_4+b_1 & b_4+b_1+b_2 \\ c_1+c_2+c_3 & c_2+c_3+c_4 & c_3+c_4+c_1 & c_4+c_1+c_2 \\ d_1+d_2+d_3 & d_2+d_3+d_4 & d_3+d_4+d_1 & d_4+d_1+d_2 \end{vmatrix}$

$= 3\,|a_1b_2c_3d_4|$.

16. $$\frac{1}{\sin^2 A}\begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix} = a^2-(b^2+c^2-2bc\cos A).$$

17. $$\begin{vmatrix} a+b+nc & (n-1)a & (n-1)b \\ (n-1)c & b+c+na & (n-1)b \\ (n-1)c & (n-1)a & c+a+nb \end{vmatrix} = n(a+b+c)^3.$$

18. $$\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

19. What is the coefficient of a_{34} in $|a_{15}|$?

20. From the first five rows of $|a_1 b_2 c_3 d_4 e_5 f_6 g_7 h_8|$ write all the possible minors that can be formed, and their complementaries.

How many minors, each a determinant of the kth order, can be formed from any k rows of $|a_{1n}|$?

21. If each of the elements of any line is the sum of the corresponding elements of two or more parallel lines, multiplied respectively by constant factors, the determinant vanishes.

22. Show that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ x_1 & a_1 & b_1 & c_1 \\ x_2 & a_2 & b_2 & c_2 \\ x_3 & a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & y_1 \\ a_2 & b_2 & c_2 & y_2 \\ a_3 & b_3 & c_3 & y_3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & u_1 & v_1 \\ a_2 & b_2 & c_2 & u_2 & v_2 \\ a_3 & b_3 & c_3 & u_3 & v_3 \\ 0 & 0 & 0 & 1 & v_4 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

From this example it appears *that any determinant may be expressed as a determinant of higher order by writing a zero above every column, prefixing a* 1 *to the row of zeros thus formed, and filling in the new column having* 1 *at the top with any n finite quantities.*

23. If in any determinant each element of the first row is unity, and if each element of every other row is the sum of the elements above and to the left of it in the preceding row, commencing with the element directly above, the determinant equals 1.

24. Any determinant of order n, in which one element is zero, is equal to the product of two factors, one of which is a determinant of the nth order, in which every other element of the row and column containing the zero is unity.

25. If in any determinant the first element is zero, and if each of the remaining elements in the first row and first column is unity, the determinant is unchanged when each element of the minor corresponding to the zero element is increased or diminished by the same quantity.

26. A determinant of the nth order is expressible as the sum of n determinants, the first of which is obtained by changing into zero each element of any line except the first element, the second by changing into zero the elements of the same line except the second element, and so on.

27. If in two determinants Δ, Δ' of the nth order, the first row of Δ is the last row of Δ', the second row of Δ the $(n-1)$th row of Δ', the third row of Δ the $(n-2)$th row of Δ', and so on; then

$$\Delta = (-1)^{\frac{n(n-1)}{2}}\Delta'.$$

28. If in two determinants Δ, Δ' of the nth order, the first row of Δ when reversed is the last row of Δ', the second row of Δ when reversed is the $(n-1)$th row of Δ', the third row of Δ when reversed is the $(n-2)$th row of Δ', etc.; then $\Delta = \Delta'$.

45. Theorem. — *If the elements of any line in a determinant are respectively multiplied by the complementary minors taken alternately plus and minus (i.e., the co-factors) of the corresponding elements of any parallel line, the sum of the products is zero.*

Consider the two determinants,

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & \dots & l_1 \\ a_2 & b_2 & c_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_k & b_k & c_k & \dots & l_k \\ \dots & \dots & \dots & \dots & \dots \\ a_p & b_p & c_p & \dots & l_p \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & l_n \end{vmatrix} \text{ and } \Delta' \equiv \begin{vmatrix} a_1 & b_1 & c_1 & \dots & l_1 \\ a_2 & b_2 & c_2 & \dots & l_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_k & b_k & c_k & \dots & l_k \\ \dots & \dots & \dots & \dots & \dots \\ a_k & b_k & c_k & \dots & l_k \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & l_n \end{vmatrix},$$

where Δ' differs from Δ only in having the kth and pth rows identical. Employing the notation of **39**, and expanding in terms of the elements of the pth row,

$$\Delta = a_p A_p + b_p B_p + c_p C_p + \cdots + l_p L_p;$$

$$\Delta' = a_k A_p + b_k B_p + c_k C_p + \cdots + l_k L_p = 0.$$

Comparing these two expansions, we observe that the second may be obtained from the first by substituting for the elements of the pth row of Δ the elements of the kth row; that is to say, if the elements of the kth row of Δ are multiplied by the co-factors of the corresponding elements of the pth row, the result is Δ'; since $\Delta' = 0$, the proposition is established.

Illustrations:

If in $(a_1 b_2 c_3) = a_1 (b_2 c_3) - a_2 (b_1 c_3) + a_3 (b_1 c_2)$ we multiply the elements of the second column respectively by the complementary minors of a_1, a_2, a_3, there results

$$b_1 (b_2 c_3 - b_3 c_2) - b_2 (b_1 c_3 - b_3 c_1) + b_3 (b_1 c_2 - b_2 c_1) = 0.$$

Let the student prove the proposition, using a determinant of the fourth order.

46. A determinant is said to be *zero-axial* if each element of the principal diagonal is zero. Thus the following are zero-axial determinants:

$$\begin{vmatrix} 0 & b_1 & c_1 \\ a_2 & 0 & c_2 \\ a_3 & b_3 & 0 \end{vmatrix}; \quad \begin{vmatrix} 0 & b_1 & c_1 & d_1 \\ a_2 & 0 & c_2 & d_2 \\ a_3 & b_3 & 0 & d_3 \\ a_4 & b_4 & c_4 & 0 \end{vmatrix}.$$

47. THEOREM. — *Any determinant may be decomposed into a sum of zero-axial determinants: the first of these is obtained by substituting zero for each element of the principal diagonal of the given determinant; the next n, by multiplying each element of the principal diagonal by its complementary minor made zero-axial; the next $\frac{n}{2}(n-1)$, by multiplying each product of pairs of elements of the principal diagonal by its complementary minor made zero-axial, and so on.*

In $\Delta^{(n)} \equiv |a_1 b_2 c_3 \dots l_n|$ change the elements of the principal diagonal into zeros, and let $\Delta_0^{(n)}$ denote the resulting determinant. Whence,

$$\Delta_0^{(n)} \equiv \begin{vmatrix} 0 & b_1 & c_1 & \dots & l_1 \\ a_2 & 0 & c_2 & \dots & l_2 \\ a_3 & b_3 & 0 & \dots & l_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & 0 \end{vmatrix}.$$

Let $\Delta_0^{(n-1)}$ denote the minor of $\Delta_0^{(n)}$ obtained by suppressing any one row of $\Delta_0^{(n)}$; $\Delta_0^{(n-2)}$ denote the minor obtained by suppressing any two rows of $\Delta_0^{(n)}$; and, in general, let $\Delta_0^{(n-i)}$ denote the minor obtained by suppressing any i rows of $\Delta_0^{(n)}$. Also let C_2 denote any product of the elements of the principal diagonal of $\Delta^{(n)}$ taken 2 and 2; C_3 any product of those elements taken 3 and 3; and, in general, C_i any product of the elements of the principal diagonal of $\Delta^{(n)}$ taken i and i. Now, $\Delta_0^{(n)}$ evidently contains all those terms of $\Delta^{(n)}$ which involve no element from the principal diagonal. $C_1\Delta_0^{(n-1)}$ must be one of those terms of the series which involve only a single element from the principal diagonal of $\Delta^{(n)}$; consequently $\Sigma C_1\Delta_0^{(n-1)}$ will be the sum of all the terms that contain only one element from the principal diagonal of $\Delta^{(n)}$. Similarly, $\Sigma C_2\Delta_0^{(n-2)}$ will be the sum of all the terms that contain only two of those elements. And, in general, $\Sigma C_i\Delta_0^{(n-i)}$ will be the sum of all the terms containing i elements of the principal diagonal of $\Delta^{(n)}$. Whence,

$$\Delta^{(n)} = \Delta_0^{(n)} + \Sigma C_1\Delta_0^{(n-1)} + \Sigma C_2\Delta_0^{(n-2)} + \Sigma C_3\Delta_0^{(n-3)} + \dots + \Sigma C_i\Delta_0^{(n-i)} + \dots + \Sigma C_{n-2}\Delta_0^{(2)} + C_n.$$

It is to be noticed that $\Delta_0^{(1)} = 0$; *i.e.*, there is a break in the series, — there being no term containing only $n-1$ of the elements in the principal diagonal.

Illustration:

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} 0 & y_1 & z_1 \\ x_2 & 0 & z_2 \\ x_3 & y_3 & 0 \end{vmatrix} + x_1\begin{vmatrix} 0 & z_2 \\ y_3 & 0 \end{vmatrix} + y_2\begin{vmatrix} 0 & z_1 \\ x_3 & 0 \end{vmatrix} + z_3\begin{vmatrix} 0 & y_1 \\ x_2 & 0 \end{vmatrix} + x_1y_2z_3.$$

48. THEOREM.—*If each consecutive pair of elements in the first row of a determinant Δ is taken with each pair of corresponding elements of the other consecutive rows to form determinants of the second degree, and if these determinants of the second degree are used in order as the elements of a new determinant Δ', then Δ equals Δ' divided by the product of all the elements except the first and last in the first row of Δ.*

We are to show that

$$\begin{vmatrix} a_1 & b_1 & c_1 & \dots & r_1 & l_1 \\ a_2 & b_2 & c_2 & \dots & r_2 & l_2 \\ a_3 & b_3 & c_3 & \dots & r_3 & l_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & r_n & l_n \end{vmatrix} = \frac{1}{b_1 c_1 d_1 \dots r_1} \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & \dots & \begin{vmatrix} r_1 & l_1 \\ r_2 & l_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} & \dots & \begin{vmatrix} r_1 & l_1 \\ r_3 & l_3 \end{vmatrix} \\ \dots & \dots & \dots & \dots \\ \begin{vmatrix} a_1 & b_1 \\ a_n & b_n \end{vmatrix} & \begin{vmatrix} b_1 & c_1 \\ b_n & c_n \end{vmatrix} & \dots & \begin{vmatrix} r_1 & l_1 \\ r_n & l_n \end{vmatrix} \end{vmatrix};$$

calling the first determinant Δ, and the second Δ'. Multiplying the first column of Δ by $-b_1$, and the second column by a_1, and adding, there results

$$-b_1\Delta = \begin{vmatrix} 0 & b_1 & c_1 & \dots & r_1 & l_1 \\ -a_2b_1 + a_1b_2 & b_2 & c_2 & \dots & r_2 & l_2 \\ -a_3b_1 + a_1b_3 & b_3 & c_3 & \dots & r_3 & l_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -a_nb_1 + a_1b_n & b_n & c_n & \dots & r_n & l_n \end{vmatrix}.$$

Now, multiplying the second column by $-c_1$, and the third column by b_1, and adding, we have

$$b_1c_1\Delta = \begin{vmatrix} 0 & 0 & c_1 & \dots & r_1 & l_1 \\ -a_2b_1 + a_1b_2 & -b_2c_1 + b_1c_2 & c_2 & \dots & r_2 & l_2 \\ -a_3b_1 + a_1b_3 & -b_3c_1 + b_1c_3 & c_3 & \dots & r_3 & l_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -a_nb_1 + a_1b_n & -b_nc_1 + b_1c_n & c_n & \dots & r_n & l_n \end{vmatrix}.$$

Proceeding in a similar manner, we have, after $(n-1)$ transformations,

$$(-1)^{n-1}b_1c_1d_1 \dots l_1\Delta$$

$$= \begin{vmatrix} 0 & 0 & 0 & \dots & 0 & l_1 \\ -a_2b_1 + a_1b_2 & -b_2c_1 + b_1c_2 & -c_2d_1 + c_1d_2 & \dots & -r_2l_1 + r_1l_2 & l_2 \\ -a_3b_1 + a_1b_3 & -b_3c_1 + b_1c_3 & -c_3d_1 + c_1d_3 & \dots & -r_3l_1 + r_1l_3 & l_3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ -a_nb_1 + a_1b_n & -b_nc_1 + b_1c_n & -c_nd_1 + c_1d_n & \dots & -r_nl_1 + r_1l_n & l_n \end{vmatrix}.$$

Now, applying **43**, and dividing by $(-1)^{n-1} b_1 c_1 d_1 \dots l_1$, we have

$$\Delta = \begin{vmatrix} |a_1 \ b_2| & |b_1 \ c_2| & \dots & |r_1 \ l_2| \\ |a_1 \ b_3| & |b_1 \ c_3| & \dots & |r_1 \ l_3| \\ \dots & \dots & \dots & \dots \\ |a_1 \ b_n| & |b_1 \ c_n| & \dots & |r_1 \ l_n| \end{vmatrix} \div b_1 c_1 d_1 \dots r_1,$$

which establishes the proposition.

49. Since by the preceding proposition any determinant of the nth order may be reduced to one of the $(n-1)$th order, we have another means of simplifying any given determinant. The proposition is especially advantageous in the reduction of determinants whose elements are given numbers. Thus:

$$\Delta \equiv \begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \\ 1 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 \end{vmatrix} = \frac{1}{6} \begin{vmatrix} -4 & -4 & 8 \\ 1 & -1 & -1 \\ -6 & -6 & -6 \end{vmatrix} = -4 \begin{vmatrix} -1 & -1 & 2 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & 3 \\ 0 & -3 \end{vmatrix} = -24.$$

Here we can mentally reduce the determinants of the second order obtained by combining the first pair of elements of the first row with the corresponding elements of the other rows, and obtain the elements of the first column of the new determinant, thus: $1 \times 2 - 3 \times 2 = -4$; $1 \times 3 - 1 \times 2 = 1$; $1 \times 4 - 5 \times 2 = -6$. For the elements of the second column we have similarly: $2 \times 1 - 2 \times 3 = -4$; $2 \times 4 - 3 \times 3 = -1$; $2 \times 3 - 4 \times 3 = -6$; and so on.

Let the student apply the proposition to show that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+x & 1 & 1 \\ 1 & 1 & 1+y & 1 \\ 1 & 1 & 1 & 1+z \end{vmatrix} = xyz; \text{ also } \begin{vmatrix} 10 & 4 & 17 & 13 \\ 4 & 2 & 8 & 6 \\ 3 & -1 & 8 & 1 \\ 7 & 5 & 20 & 17 \end{vmatrix} = 124.$$

Also apply the proposition to show that

$$\begin{vmatrix} 5 & 0 & 11 & 0 \\ 8 & 7 & 2 & 20 \\ 3 & 1 & 4 & 7 \\ 8 & 1 & 0 & 6 \end{vmatrix} = 2188.$$

MISCELLANEOUS EXAMPLES.

1. Find the value of

$$\begin{vmatrix} 2 & 4 & 3 & 1 & 4 & 3 \\ -4 & 2 & -3 & 2 & -1 & 2 \\ 5 & -1 & 6 & 2 & -1 & 5 \\ 1 & 1 & 1 & -2 & -2 & -2 \\ 7 & -3 & -5 & 1 & 4 & 2 \\ 3 & 1 & 2 & -1 & 2 & 3 \end{vmatrix}; \text{ also of } \begin{vmatrix} 12 & 22 & 14 & 17 & 20 & 10 \\ 16 & -4 & 7 & 1 & -2 & 15 \\ 10 & -3 & -2 & 3 & -2 & 8 \\ 7 & 12 & 8 & 9 & 11 & 6 \\ 11 & 2 & 4 & -8 & 1 & 9 \\ 24 & 6 & 6 & 3 & 4 & 22 \end{vmatrix}.$$

2. Expand the following:

$$\begin{vmatrix} x & 0 & 0 & 0 & \ldots & 0 & a_n \\ -1 & x & x^2 & x^3 & \ldots & x^{n-1} & a_{n-1} \\ 0 & -1 & 0 & 0 & \ldots & 0 & a_{n-2} \\ 0 & 0 & -1 & 0 & \ldots & 0 & a_{n-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & 0 & a_1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & a_0 \end{vmatrix}; \begin{vmatrix} a_1 & b_2 & 0 & 0 \ldots & 0 & 0 \\ a_2 & -b_1 & b_3 & 0 \ldots & 0 & 0 \\ a_3 & 0 & -b_2 & b_4 \ldots & 0 & 0 \\ a_4 & 0 & 0 & -b_3 \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots & \ldots \\ a_n & 0 & 0 & 0 \ldots & -b_{n-1} & b_{n+1} \\ a_{n+1} & 0 & 0 & 0 \ldots & 0 & -b_n \end{vmatrix}.$$

3. Show that

$$\begin{vmatrix} 1 & 0 & 0 & 0 & ax+hy+gz \\ 0 & 1 & 0 & 0 & hx+by+fz \\ 0 & 0 & 1 & 0 & gx+fy+cz \\ 0 & 0 & 0 & 1 & lx+my+nz \\ x & y & z & 1 & k \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & ax+hy+gz+l \\ 0 & 1 & 0 & hx+by+fz+m \\ 0 & 0 & 1 & gx+fy+cz+n \\ x & y & z & k \end{vmatrix}.$$

4. Write the complementaries of the following minors of

$|a_0\, b_1\, c_2\, d_3\, e_4\, f_5|$: $|c_2\, e_4|$; $|c_0\, f_5|$; $|c_0\, d_4\, e_2|$; $|b_1\, c_3\, d_4|$; $|d_2\, e_3\, f_4|$.

5. What are the complementaries of

$|a_{12}\ a_{35}|$ and $|a_{12}\ a_{35}\ a_{56}|$, in $|a_{01}\ a_{12}\ a_{23}\ a_{34}\ a_{45}\ a_{56}|$?

6. Show that

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1+a & 1+b & 1+c \\ 1 & a+1 & 0 & a+b & a+c \\ 1 & b+1 & b+a & 0 & b+c \\ 1 & c+1 & c+a & c+b & 0 \end{vmatrix} = 2^3 \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}.$$

7. Prove that

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & \dots & a_n \\ b_1 & b_2 & b_3 & b_4 & \dots & b_n \\ c_1 & c_2 & c_3 & c_4 & \dots & c_n \\ d_1 & d_2 & d_3 & d_4 & \dots & d_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ l_1 & l_2 & l_3 & l_4 & \dots & l_n \end{vmatrix} = \frac{1}{a_1^{n-2}} \begin{vmatrix} |a_1\ b_2| & |a_1\ b_3| & |a_1\ b_4| \dots & |a_1\ b_n| \\ |a_1\ c_2| & |a_1\ c_3| & |a_1\ c_4| \dots & |a_1\ c_n| \\ |a_1\ d_2| & |a_1\ d_3| & |a_1\ d_4| \dots & |a_1\ d_n| \\ \dots & \dots & \dots \quad \dots & \dots \\ |a_1\ l_2| & |a_1\ l_3| & |a_1\ l_4| \dots & |a_1\ l_n| \end{vmatrix}.$$

8. Employing the notation

$$\binom{n}{r} \equiv \frac{n(n-1)(n-2)\ \dots\ (n-r+1)}{r!},$$

show that

$$\begin{vmatrix} 1 & \binom{x+y}{1} & \binom{x+y+1}{2} & \binom{x+y+2}{3} & \dots & \binom{x+2y-1}{y} \\ 1 & \binom{x+y+1}{1} & \binom{x+y+2}{2} & \binom{x+y+3}{3} & \dots & \binom{x+2y}{y} \\ 1 & \binom{x+y+2}{1} & \binom{x+y+3}{2} & \binom{x+y+4}{3} & \dots & \binom{x+2y+1}{y} \\ & \dots & \dots & \dots & \dots & \dots \\ 1 & \binom{x+2y}{1} & \binom{x+2y+1}{2} & \binom{x+2y+2}{3} & \dots & \binom{x+3y-1}{y} \end{vmatrix} = 1.$$

Observe that $\binom{z+u}{r} - \binom{z+u-1}{r} = \binom{z+u-1}{r-1}$.

The Product of Two Determinants.

50. If we note a determinant by K, and another by L, their product P is evidently expressed by

$$\begin{vmatrix} K & \alpha \\ 0 & L \end{vmatrix}.$$

The form of this product suggests the probability that the product of two determinants may be expressed by writing the factors as complementary minors of a determinant of higher order, and filling in the vacant places due to one or both of

the factors with zeros. Suppose, for example, that K is of the third order, and L of the second; then P would take the form:

$$P = \begin{vmatrix} a_1 & b_1 & c_1 & & \\ a_2 & b_2 & c_2 & & a \\ a_3 & b_3 & c_3 & & \\ & 0 & & \alpha_4 & \beta_4 \\ & & & \alpha_5 & \beta_5 \end{vmatrix}.$$

We now wish to discover if, when we fill in the vacant places due to K or L with zeros, and thus make P a determinant of the fifth order, P will still be the product of K and L. That this is the fact will be shown in the next article.

51. Theorem. — *The product of two determinants, K and L, of degree m and n, respectively, is a determinant P, of degree $m+n$, in which K and L are complementary minors, so situated that the principal diagonal of P is made up of the elements in order of the principal diagonals of K and L; the vacant places in P, due to either K or L, are filled with zeros, and any mn finite elements occupy the remaining places.*

We have to show, for example, that

$$K \times L \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_4 & \beta_4 & \gamma_4 \\ \alpha_5 & \beta_5 & \gamma_5 \\ \alpha_6 & \beta_6 & \gamma_6 \end{vmatrix} = \left|\begin{array}{ccc|ccc} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ \hline a_4 & b_4 & c_4 & \alpha_4 & \beta_4 & \gamma_4 \\ a_5 & b_5 & c_5 & \alpha_5 & \beta_5 & \gamma_5 \\ a_6 & b_6 & c_6 & \alpha_6 & \beta_6 & \gamma_6 \end{array}\right| \equiv P. \qquad (1)$$

Developing P in terms of the elements of the fourth column and their complementary minors, we have

$$P = \alpha_4 \begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 \\ a_5 & b_5 & c_5 & \beta_5 & \gamma_5 \\ a_6 & b_6 & c_6 & \beta_6 & \gamma_6 \end{vmatrix} - \alpha_5 \begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 \\ a_4 & b_4 & c_4 & \beta_4 & \gamma_4 \\ a_6 & b_6 & c_6 & \beta_6 & \gamma_6 \end{vmatrix} + \alpha_6 \begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 \\ a_4 & b_4 & c_4 & \beta_4 & \gamma_4 \\ a_5 & b_5 & c_5 & \beta_5 & \gamma_5 \end{vmatrix},$$

or

$$P = \alpha_4 \Delta_{\alpha_4} - \alpha_5 \Delta_{\alpha_5} + \alpha_6 \Delta_{\alpha_6}. \qquad (2)$$

But $$\Delta_{\alpha_4} = \beta_5 \begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \\ a_6 & b_6 & c_6 & \gamma_6 \end{vmatrix} - \beta_6 \begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \\ a_5 & b_5 & c_5 & \gamma_5 \end{vmatrix}$$

$$= \beta_5 \gamma_6 | a_1 \; b_2 \; c_3 | - \beta_6 \gamma_5 | a_1 \; b_2 \; c_3 | \equiv K \begin{vmatrix} \beta_5 & \gamma_5 \\ \beta_6 & \gamma_6 \end{vmatrix}.$$

In the same manner, we may show that

$$\Delta_{\alpha_5} = K \begin{vmatrix} \beta_4 & \gamma_4 \\ \beta_6 & \gamma_6 \end{vmatrix}; \text{ also } \Delta_{\alpha_6} = K \begin{vmatrix} \beta_4 & \gamma_4 \\ \beta_5 & \gamma_5 \end{vmatrix}.$$

Substituting these values in (2), we have

$$P = K \{ \alpha_4 | \beta_5 \gamma_6 | - \alpha_5 | \beta_4 \gamma_6 | + \alpha_6 | \beta_4 \gamma_5 | \} = K \times L,$$

since the second factor is obviously L, expanded in terms of the elements of the first column.

The method of proof here given is perfectly general, and is applicable to determinants of any order. Thus, if in (1) we make $\gamma_4 = \gamma_5 = 0$, and $\gamma_6 = 1$, P takes the form considered in **50**. The student can readily make the application.

As another exercise, the student may show that

$$\Delta \equiv \begin{vmatrix} a_1 & 0 & c_1 & d_1 & 0 & f_1 & 0 \\ a_2 & 0 & c_2 & d_2 & 0 & f_2 & 0 \\ a_3 & 0 & c_3 & d_3 & 0 & f_3 & 0 \\ a_4 & 0 & c_4 & d_4 & 0 & f_4 & 0 \\ 0 & b_5 & 0 & 0 & e_5 & 0 & g_5 \\ 0 & b_6 & 0 & 0 & e_6 & 0 & g_6 \\ 0 & b_7 & 0 & 0 & e_7 & 0 & g_7 \end{vmatrix} = -| a_1 f_2 c_3 d_4 | \times | e_5 b_6 g_7 |.$$

What difference would it make in the result if the zeros in the fifth, sixth, and seventh rows of Δ were replaced by any finite elements?

52. Writing the product of $| a_1 b_2 c_3 |$ and $| x_1 y_2 z_3 |$, in accordance with **51**,

$$\begin{vmatrix} a_1 & b_1 & c_1 & -1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & -1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & -1 \\ 0 & 0 & 0 & x_1 & y_1 & z_1 \\ 0 & 0 & 0 & x_2 & y_2 & z_2 \\ 0 & 0 & 0 & x_3 & y_3 & z_3 \end{vmatrix} \equiv P,$$

we have, by **37**,

$$P = \begin{vmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ a_1x_1 + a_2y_1 + a_3z_1 & b_1x_1 + b_2y_1 + b_3z_1 & c_1x_1 + c_2y_1 + c_3z_1 & x_1 & y_1 & z_1 \\ a_1x_2 + a_2y_2 + a_3z_2 & b_1x_2 + b_2y_2 + b_3z_2 & c_1x_2 + c_2y_2 + c_3z_2 & x_2 & y_2 & z_2 \\ a_1x_3 + a_2y_3 + a_3z_3 & b_1x_3 + b_2y_3 + b_3z_3 & c_1x_3 + c_2y_3 + c_3z_3 & x & y_3 & z_3 \end{vmatrix},$$

which, by **43**,

$$= \begin{vmatrix} a_1x_1 + a_2y_1 + a_3z_1 & b_1x_1 + b_2y_1 + b_3z_1 & c_1x_1 + c_2y_1 + c_3z_1 \\ a_1x_2 + a_2y_2 + a_3z_2 & b_1x_2 + b_2y_2 + b_3z_2 & c_1x_2 + c_2y_2 + c_3z_2 \\ a_1x_3 + a_2y_3 + a_3z_3 & b_1x_3 + b_2y_3 + b_3z_3 & c_1x_3 + c_2y_3 + c_3z_3 \end{vmatrix}.$$

This result expresses the product of two determinants of the third order as a determinant of the same order. We are thus led to infer that the product of two determinants of any order may be expressed at once as a determinant of the same order.

We now proceed to establish this important multiplication theorem.

53. THEOREM. — *The product of two determinants,* Δ, Δ' *of the nth order is a determinant* Δ'' *of the same order. Any element* a_{rs} *of* Δ'' *is obtained by multiplying each element of the rth row of* Δ *by the corresponding element of the sth row of* Δ', *and adding the products.**

Before giving the general demonstration, it will be useful to establish the proposition for the product of two determinants of the third order, and note carefully the form of the result.

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* Forming the product by columns, the statement is, of course: The element in the rth column and sth row of Δ'' is obtained by multiplying each element in the rth column of Δ by the corresponding element in the sth column of Δ', and adding the products.

Put $\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $\Delta' \equiv \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$.

Applying the theorem, we have to show that

$$\Delta\Delta' = \Delta'' \equiv \begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}.$$

Since each element of Δ'' is a trinomial, the determinant may be decomposed into twenty-seven determinants (**35**), the elements of which will be monomials. But of these twenty-seven determinants only six do not vanish.* Those determinants which do not vanish are formed by taking for the first column a set of first terms from the first column of Δ'', for the second column a set of second terms from the second column of Δ'', for the third column a set of third terms from the third column; or, by taking a set of second terms from the first column of Δ'', a set of first terms from the second column of Δ'', and a set of third terms from the third column of Δ''; and so on. That is to say, exactly as many non-vanishing determinants can be formed from Δ'' as there are permutations of the numbers 1, 2, 3, *i.e.*, 6. Hence

$$\Delta'' = \begin{vmatrix} a_1\alpha_1 & b_1\beta_2 & c_1\gamma_3 \\ a_2\alpha_1 & b_2\beta_2 & c_2\gamma_3 \\ a_3\alpha_1 & b_3\beta_2 & c_3\gamma_3 \end{vmatrix} + \begin{vmatrix} a_1\alpha_1 & c_1\gamma_2 & b_1\beta_3 \\ a_2\alpha_1 & c_2\gamma_2 & b_2\beta_3 \\ a_3\alpha_1 & c_3\gamma_2 & b_3\beta_3 \end{vmatrix} + \begin{vmatrix} b_1\beta_1 & a_1\alpha_2 & c_1\gamma_3 \\ b_2\beta_1 & a_2\alpha_2 & c_2\gamma_3 \\ b_3\beta_1 & a_3\alpha_2 & c_3\gamma_3 \end{vmatrix}$$

$$+ \begin{vmatrix} b_1\beta_1 & c_1\gamma_2 & a_1\alpha_3 \\ b_2\beta_1 & c_2\gamma_2 & a_2\alpha_3 \\ b_3\beta_1 & c_3\gamma_2 & a_3\alpha_3 \end{vmatrix} + \begin{vmatrix} c_1\gamma_1 & b_1\beta_2 & a_1\alpha_3 \\ c_2\gamma_1 & b_2\beta_2 & a_2\alpha_3 \\ c_3\gamma_1 & b_3\beta_2 & a_3\alpha_3 \end{vmatrix} + \begin{vmatrix} c_1\gamma_1 & a_1\alpha_2 & b_1\beta_3 \\ c_2\gamma_1 & a_2\alpha_2 & b_2\beta_3 \\ c_3\gamma_1 & a_3\alpha_2 & b_3\beta_3 \end{vmatrix}$$

$$= \alpha_1\beta_2\gamma_3(a_1b_2c_3) - \alpha_1\beta_3\gamma_2(a_1b_2c_3) - \alpha_2\beta_1\gamma_3(a_1b_2c_3)$$

$$+ \alpha_3\beta_1\gamma_2(a_1b_2c_3) - \alpha_3\beta_2\gamma_1(a_1b_2c_3) + \alpha_2\beta_3\gamma_1(a_1b_2c_3)$$

(by **30** and **31**)

---

* It is obvious that the determinants formed from sets of first terms taken from the three columns of Δ'', or those containing sets of first terms from two columns, etc., must vanish. Similarly for determinants formed from sets of second terms, and so on.

$$= (a_1b_2c_3)\left[\alpha_1\beta_2\gamma_3+\alpha_2\beta_3\gamma_1+\alpha_3\beta_1\gamma_2-\alpha_3\beta_2\gamma_1-\alpha_2\beta_1\gamma_3-\alpha_1\beta_3\gamma_2\right]$$
$$= (a_1b_2c_3)(\alpha_1\beta_2\gamma_3),$$

which establishes the proposition for the special case under consideration.

In general, let

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & \dots & l_1 \\ a_2 & b_2 & c_2 & \dots & l_2 \\ a_3 & b_3 & c_3 & \dots & l_3 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & \dots & l_n \end{vmatrix} \text{ and } \Delta' \equiv \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 & \dots & \lambda_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & \lambda_2 \\ \alpha_3 & \beta_3 & \gamma_3 & \dots & \lambda_3 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_n & \beta_n & \gamma_n & \dots & \lambda_n \end{vmatrix}.$$

Then the product $\Delta\Delta' = \Delta''$

$$\equiv \begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1+\dots+l_1\lambda_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2+\dots+l_1\lambda_2 & \dots & a_1\alpha_n+b_1\beta_n+c_1\gamma_n+\dots+l_1\lambda_n \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1+\dots+l_2\lambda_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2+\dots+l_2\lambda_2 & \dots & a_2\alpha_n+b_2\beta_n+c_2\gamma_n+\dots+l_2\lambda_n \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1+\dots+l_3\lambda_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2+\dots+l_3\lambda_2 & \dots & a_3\alpha_n+b_3\beta_n+c_3\gamma_n+\dots+l_3\lambda_n \\ \dots\ \ \dots\ \ \dots\ \ \dots\ \ \dots & \dots\ \ \dots\ \ \dots\ \ \dots\ \ \dots & \dots & \dots\ \ \dots\ \ \dots\ \ \dots\ \ \dots \\ a_n\alpha_1+b_n\beta_1+c_n\gamma_1+\dots+l_n\lambda_1 & a_n\alpha_2+b_n\beta_2+c_n\gamma_2+\dots+l_n\;{}_2 & \dots & a_n\alpha_n+b_n\beta_n+c_n\gamma_n+\dots+l_n\lambda_n \end{vmatrix}.$$

Now, Δ'' may be decomposed into a sum of n^n determinants, the elements of which are monomials. But it is obvious that all those determinants whose columns are formed from sets of first terms of the columns of Δ'', or from sets of second terms, etc., will vanish, as each will contain identical columns. In fact, all those determinants into which Δ'' is decomposed will vanish that have not the first column formed from a set of kth terms from the first column of Δ'', the second column formed from a set of rth terms from the second column of Δ'', the third column formed from a set of tth terms from the third column of Δ'', and so on. Now, as many such non-vanishing determinants can be formed as there are permutations of the numbers 1, 2, 3 ... n; that is, $n\,!$ Hence, Δ'' is decomposable into $n\,!$ determinants, of which the following Δ_r is the type:

$$\Delta_r \equiv \begin{vmatrix} b_1\beta_1 & l_1\lambda_2 & a_1\alpha_3 & \dots & c_1\gamma_n \\ b_2\beta_1 & l_2\lambda_2 & a_2\alpha_3 & \dots & c_2\gamma_n \\ b_3\beta_1 & l_3\lambda_2 & a_3\alpha_3 & \dots & c_3\gamma_n \\ \dots & \dots & \dots & \dots & \dots \\ b_n\beta_1 & l_n\lambda_2 & a_n\alpha_3 & \dots & c_n\gamma_n \end{vmatrix}$$

But $\Delta_r = \beta_1\lambda_2\alpha_3 \dots \gamma_n \,|\, b_1 l_2 a_3 \dots c_n \,|$.

Now, the determinant factor of Δ_r is evidently Δ multiplied by the sign-factor $(-1)^p$, in which p is the number of interchanges of two columns which must be made in Δ to leave its columns in the order which they have in Δ_r. Accordingly, $\Delta_r = (-1)^p \beta_1\lambda_2\alpha_3 \dots \gamma_n \Delta$. But $(-1)^p \beta_1\lambda_2\alpha_3 \dots \gamma_n$ is a term of Δ', since the number of interchanges of two letters which must be made in $\alpha_1\beta_2\gamma_3 \dots \lambda_n$ to obtain the arrangement here given is p. Accordingly, Δ_r equals a term of Δ' multiplied by Δ. Thus each of the $n!$ determinants into which Δ'' has been decomposed is the product of Δ, and a term of Δ'. $\therefore \Delta'' = \Delta\Delta'$.

Illustrations:

$$\begin{vmatrix} 1 & 2 & 0 & 3 \\ 1 & 1 & 1 & 0 \\ 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{vmatrix} \times \begin{vmatrix} 0 & 1 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0+2+0+0 & 3+4+0+3 & 1+0+0+3 & 2+2+0+0 \\ 0+1+1+0 & 3+2+1+0 & 1+0+0+0 & 2+1+1+0 \\ 0+0+2+0 & 9+0+2+1 & 3+0+0+1 & 6+0+2+0 \\ 0+0+1+0 & 0+0+1+2 & 0+0+0+2 & 0+0+1+0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 10 & 4 & 4 \\ 2 & 6 & 1 & 4 \\ 2 & 12 & 4 & 8 \\ 1 & 3 & 2 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & 5 & 2 & 2 \\ 2 & 6 & 1 & 4 \\ 1 & 6 & 2 & 4 \\ 1 & 3 & 2 & 1 \end{vmatrix}.$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & \alpha \\ 0 & 1 & b & \beta \\ 0 & 1 & c & \gamma \end{vmatrix} \times \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & a & \alpha \\ 1 & 0 & b & \beta \\ 1 & 0 & c & \gamma \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & a^2+\alpha^2 & ab+\alpha\beta & ac+\alpha\gamma \\ 1 & ab+\alpha\beta & b^2+\beta^2 & bc+\beta\gamma \\ 1 & ac+\alpha\gamma & bc+\beta\gamma & c^2+\gamma^2 \end{vmatrix}.$$

54. Since, before multiplying two determinants together, we may change the form of one or both factors, the product of two determinants can be expressed in a variety of different forms. As an illustration, the student may verify the following equations:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + a_2\beta_1 & a_1\alpha_2 + a_2\beta_2 \\ b_1\alpha_1 + b_2\beta_1 & b_1\alpha_2 + b_2\beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + b_1\alpha_2 & a_1\beta_1 + b_1\beta_2 \\ a_2\alpha_1 + b_2\alpha_2 & a_2\beta_1 + b_2\beta_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1\alpha_1 + a_2\alpha_2 & a_1\beta_1 + a_2\beta_2 \\ b_1\alpha_1 + b_2\alpha_2 & b_1\beta_1 + b_2\beta_2 \end{vmatrix}.$$

EXAMPLES.

1. Show that one form of the product of

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \times \begin{vmatrix} a^2 - a & 1 \\ b^2 - b & 1 \\ c^2 - c & 1 \end{vmatrix} \text{ is } \begin{vmatrix} a^2 & a^2 - ab + b^2 & a^2 - ac + c^2 \\ a^2 - ab + b^2 & b^2 & b^2 - bc + c^2 \\ a^2 - ac + c^2 & b^2 - cb + c^2 & c^2 \end{vmatrix}.$$

2. One form of the product of

$$\begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix} \times \begin{vmatrix} a+b+\frac{1}{2}c & -\frac{1}{2}a & -\frac{1}{2}b \\ -\frac{1}{2}c & b+c+\frac{1}{2}a & -\frac{1}{2}b \\ -\frac{1}{2}c & -\frac{1}{2}a & c+a+\frac{1}{2}b \end{vmatrix} \text{ is}$$

$$\begin{vmatrix} (a+b)^2 & a^2 & b^2 \\ c^2 & (b+c)^2 & b^2 \\ c^2 & a^2 & (c+a)^2 \end{vmatrix}.$$

3. Find the product of $\begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix}$ and $\begin{vmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{vmatrix}$, and

thence show that the first determinant $= a(b-a)(c-b)(d-c)$.

4. Show that

$$\begin{vmatrix} -a & b & c & d \\ b & -a & d & c \\ c & d & -a & b \\ d & c & b & -a \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} b+c+d-a & a-b+c+d & a+b-c+d & a+b+c-d \\ b-a+d+c & -b+a+d+c & -b-a-d+c & -b-a+d-c \\ c+d-a+b & -c-d-a+b & -c+d+a+b & -c+d-a-b \\ d+c+b-a & -d-c+b-a & -d+c-b-a & -d+c+b+a \end{vmatrix}$$

$$=(b+c+d-a)\,(c+d+a-b)\,(d+a+b-c)\,(a+b+c-d)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix},$$

and thence show that the first determinant

$$=-(b+c+d-a)(c+d+a-b)(d+a+b-c)(a+b+c-d).$$

5. Show that

$$\begin{vmatrix} 1 & a^2+\alpha^2 & -2a & -2\alpha \\ 1 & b^2+\beta^2 & -2b & -2\beta \\ 1 & c^2+\gamma^2 & -2c & -2\gamma \\ 1 & d^2+\delta^2 & -2d & -2\delta \end{vmatrix} \times \begin{vmatrix} a^2+\alpha^2 & 1 & a & \alpha \\ b^2+\beta^2 & 1 & b & \beta \\ c^2+\gamma^2 & 1 & c & \gamma \\ d^2+\delta^2 & 1 & d & \delta \end{vmatrix}$$

$$=\begin{vmatrix} 0 & (a-b)^2+(\alpha-\beta)^2 & (a-c)^2+(\alpha-\gamma)^2 & (a-d)^2+(\alpha-\delta)^2 \\ (a-b)^2+(\alpha-\beta)^2 & 0 & (b-c)^2+(\beta-\gamma)^2 & (b-d)^2+(\beta-\delta)^2 \\ (a-c)^2+(\alpha-\gamma)^2 & (b-c)^2+(\beta-\gamma)^2 & 0 & (c-d)^2+(\gamma-\delta)^2 \\ (a-d)^2+(\alpha-\delta)^2 & (b-d)^2+(\beta-\delta)^2 & (c-d)^2+(\gamma-\delta)^2 & 0 \end{vmatrix}.$$

6. Show that

$$\begin{vmatrix} a_1 & b_1 & c_1 & 1 \\ a_2 & b_2 & c_2 & 1 \\ a_3 & b_3 & c_3 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 & k_1 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & k_3 \\ 0 & 0 & 0 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ h_1 & h_2 & h_3 & 1 \end{vmatrix}$$

equals the determinant in Example 12, page 38.

7. Find the two determinant factors of

$$\begin{vmatrix} a_1 & b_1x_1+c_1y_1 & b_1x_2+c_1y_2 \\ a_2 & b_2x_1+c_2y_1 & b_2x_2+c_2y_2 \\ a_3 & b_3x_1+c_3y_1 & b_3x_2+c_3y_2 \end{vmatrix}; \text{ also of } \begin{vmatrix} ax_1+cz_1 & 0 & fx_1+gz_1 \\ ax_2+by_2+cz_2 & dy_2 & fx_2+gz_2 \\ by_3+cz_3 & dy_3 & gz_3 \end{vmatrix}.$$

8. Form the product of $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and $\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$.

The order of the first determinant may be raised to that of the second by writing it $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{vmatrix}$, and the product can then

be found in the usual way. If we wish the product to contain only the elements found in the two factors, how should the first determinant be written?

From this example it is evident that *the product of any number of determinants of different degrees can be expressed as a determinant of the nth degree, n being the highest degree among the factors.*

9. Employing the notation $i \equiv \sqrt{-1}$, show that the product of

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} \text{ and } \begin{vmatrix} a_1-ib_1 & c_1-id_1 \\ -c_1-id_1 & a_1+ib_1 \end{vmatrix}$$

may be written
$$\begin{vmatrix} D-iC & B-iA \\ -B-iA & D+iC \end{vmatrix},$$

in which

$$A \equiv bc_1 - b_1c + ad_1 - a_1d \quad C \equiv ab_1 - a_1b + cd_1 - c_1d$$
$$B \equiv ca_1 - c_1a + bd_1 - b_1d \quad D \equiv aa_1 + bb_1 + cc_1 + dd_1;$$

and thence show that *the product of two sums, each of four squares, is itself the sum of four squares.* (Euler's Theorem.)

10. Show (1) that the product of $|a_1 b_2 c_3|$ and $|p_1 q_2 r_3|$ may be expressed as a determinant of the fourth order by writing the two factors
$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \text{ and } -\begin{vmatrix} p_1 & q_1 & 0 & r_1 \\ p_2 & q_2 & 0 & r_2 \\ p_3 & q_3 & 0 & r_3 \\ 0 & 0 & 1 & 0 \end{vmatrix}$$
respectively.

(2) By writing the two factors

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} p_1 & 0 & 0 & q_1 & r_1 \\ p_2 & 0 & 0 & q_2 & r_2 \\ p_3 & 0 & 0 & q_3 & r_3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

show that the product is a determinant of the fifth order.

(3) By writing the two factors

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & p_1 & q_1 & r_1 \\ 0 & 0 & 0 & p_2 & q_2 & r_2 \\ 0 & 0 & 0 & p_3 & q_3 & r_3 \end{vmatrix}$$

show that the product is a determinant of the sixth order. This example, and the theorems of **51** and **53**, show that *the product of two determinants of the nth order can be expressed as a determinant of each of the following orders: nth,* $(n+1)$*th,* $(n+2)$*th* $\cdots$ $(2n-1)$*th,* 2*nth.*

55. Theorem. — *Any determinant* Δ *may be expanded as a sum of products of pairs of minors. The first factor of each product is a minor of the rth degree, formed from a set of r chosen rows, and the other factor is the complementary minor of the first factor. The sign of a product is* $+$ *or* $-$*, according as the product of the principal terms of the factors regarded as a term of* Δ *is* $+$ *or* $-$.*

Every term of Δ contains r elements from the columns of a set of r columns found in the n columns and first r rows of Δ. That is to say, from every minor of the rth degree formed from the first r rows, $r!$ partial terms of Δ can be formed. Now, the remaining $(n-r)$ elements of every such partial term will be found in the remaining rows and columns after removing one of these minors of the rth degree. Or, in other words, $(n-r)!$ partial terms of Δ corresponding to the $r!$ other partial terms are found in every minor complementary to one of the first set. $\dfrac{n!}{r!\,(n-r)!}$ minors of the rth degree can be formed from the first r rows. Now, the product of two such complementary minors gives $r!\,(n-r)!$ terms of Δ; consequently, the sum of all the products gives $n!$ terms, *i.e.*, the full number of terms in Δ.

To fix the sign of any product in this expansion, we have only to remember that its sign must be the same as the sign of the product of the principal terms of the two minors. This latter product being a term of Δ, the sign of the product of the two minors must be the sign of the product of their principal terms, regarded as a term of Δ.

If the selected rows are not the first r rows, we can easily make them so; then, after giving Δ the proper sign factor, the demonstration applies as given.

* This expansion is known as Laplace's Theorem.

Illustrations:

Selecting the first two rows in $|a_1\,b_2\,c_3\,d_4|$, we have

$$|a_1\,b_2\,c_3\,d_4| = |a_1\,b_2|\,|c_3\,d_4| - |a_1\,c_2|\,|b_3\,d_4| + |a_1\,d_2|\,|b_3\,c_4| \\ + |b_1\,c_2|\,|a_3\,d_4| - |b_1\,d_2|\,|a_3\,c_4| + |c_1\,d_2|\,|a_3 b_4|.$$

Let the student select the first two columns of $|a_1\,b_2\,c_3\,d_4|$, and expand, obtaining

$$|a_1 b_2|\,|c_3 d_4| - |a_1 b_3|\,|c_2 d_4| + |a_1 b_4|\,|c_2\,d_3| + |a_2\,b_3|\,|c_1 d_4| \\ - |a_2 b_4|\,|c_1 d_3| + |a_3 b_4|\,|c_1 d_2|.$$

Show that

$$|a_1 b_2 c_3 d_4 e_5| = -|a_2 b_4|\,|c_1 d_3 e_5| + |a_2 c_4|\,|b_1 d_3 e_5| - |a_2 d_4|\,|b_1 c_3 e_5| \\ + |a_2 e_4|\,|b_1 c_3 d_5| - |b_2 c_4|\,|a_1 d_3 e_5| + |b_2 d_4|\,|a_1 c_3 e_5| \\ - |b_2 e_4|\,|a_1 c_3 d_5| + |c_2 d_4|\,|a_1 b_3\,e_5| + |c_2\,e_4|\,|a_1 b_3 d_5| \\ - |d_2 e_4|\,|a_1 b_3 c_5|.$$

What is the relation of **41** to the present theorem?

56. It will be interesting to note what results, if, instead of multiplying the minors of the rth degree formed from r chosen lines by their complementaries, as in the last article, we multiply every such minor by the complementary of a corresponding minor formed from r lines different from those first chosen. By the preceding article

$$|a_1 b_2 c_3 d_4 e_5|$$
$$= |a_1 b_2|\,|c_3 d_4 e_5| - |a_1 b_3|\,|c_2 d_4 e_5| + |a_1 b_4|\,|c_2 d_3 e_5| - |a_1 b_5|\,|c_2 d_3 e_4|$$
$$+ |a_2 b_3|\,|c_1 d_4 e_5| - |a_2 b_4|\,|c_1 d_3 e_5| + |a_2 b_5|\,|c_1 d_3 e_4| + |a_3 b_4|\,|c_1 d_2 e_5|$$
$$- |a_3 b_5|\,|c_1 d_2 e_4| + |a_4 b_5|\,|c_1 d_2 e_3|.$$

Now, if in the above we write c for b, it is evident that the determinant on the left vanishes, and hence the second member vanishes; but by this substitution we multiply the minors formed from the *first* and *third* columns of $|a_1 b_2 c_3 d_4 e_5|$ by the complementaries of the corresponding minors formed from the *first* and *second* columns. It is obvious that the truth here exemplified holds in general. Moreover, it includes the special case of **45**.

In symbols, the expansion of a determinant by **55** is expressed by writing
$$\Delta = \Sigma \mid a_p\, b_q \mid \Delta_{a_p,\, b_q},$$
where the chosen columns are two in number, or
$$\Delta = \Sigma \mid a_p\, b_q\, c_r \mid \Delta_{a_p,\, b_q,\, c_r},$$
where the chosen columns are three in number; and so on.

Employing the notation of double subscripts, we have, in general,
$$\Delta \equiv \mid a_{1n} \mid = \Sigma \mid a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_r q_r} \mid \Delta_{a_{p_1 q_1},\, a_{p_2 q_2},\, \ldots\, a_{p_r q_r}}.$$

57. Theorem. — *The product of a determinant* $\Delta \equiv |a_{1n}|$, *and any one of its minors* M, *of order* m, *is a determinant* Δ' *of order* $n + m$. Δ' *is expressible as the sum of products of pairs of minors of* Δ; *the first factor of each product is a minor of* Δ, *formed from* r *chosen rows containing* M, *and the second factor is that minor of* Δ *containing the complementary of the first factor and the minor* M. *The sign of each product is determined as in* **55**.

Let the chosen rows referred to in the statement of the theorem be the first r; then, by **51**, we have at once

$$\Delta' \equiv \left| \begin{array}{ccccccccccccccc}
a_{11} & a_{12} & \cdots & a_{1\,k-1} & a_{1k} & \cdots & a_{1\,r-1} & a_{1r} & a_{1\,r+1} & \cdots & a_{1n} & 0 & \cdots & 0 & 0 \\
a_{21} & a_{22} & \cdots & a_{2\,k-1} & a_{2k} & \cdots & a_{2\,r-1} & a_{2r} & a_{2\,r+1} & \cdots & a_{2n} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{k1} & a_{k2} & \cdots & a_{k\,k-1} & a_{kk} & \cdots & a_{k\,r-1} & a_{kr} & a_{k\,r+1} & \cdots & a_{kn} & 0 & \cdots & 0 & 0 \\
a_{k+1\,1} & a_{k+1\,2} & \cdots & a_{k+1\,k-1} & a_{k+1\,k} & \cdots & a_{k+1\,r-1} & a_{k+1\,r} & a_{k+1\,r+1} & \cdots & a_{k+1\,n} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{r-1\,1} & a_{r-1\,2} & \cdots & a_{r-1\,k-1} & a_{r-1\,k} & \cdots & a_{r-1\,r-1} & a_{r-1\,r} & a_{r-1\,r+1} & \cdots & a_{r-1\,n} & 0 & \cdots & 0 & 0 \\
a_{r1} & a_{r2} & \cdots & a_{r\,k-1} & a_{rk} & \cdots & a_{r\,r-1} & a_{rr} & a_{r\,r+1} & \cdots & a_{rn} & 0 & \cdots & 0 & 0 \\
a_{r+1\,1} & a_{r+1\,2} & \cdots & a_{r+1\,k-1} & a_{r+1\,k} & \cdots & a_{r+1\,r-1} & a_{r+1\,r} & a_{r+1\,r+1} & \cdots & a_{r+1\,n} & a_{r+1\,k} & \cdots & a_{r+1\,r-1} & a_{r+1\,r} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n1} & a_{n2} & \cdots & a_{n\,k-1} & a_{nk} & \cdots & a_{n\,r-1} & a_{nr} & a_{n\,r+1} & \cdots & a_{nn} & a_{nk} & \cdots & a_{n\,r-1} & a_{nr} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{kk} & \cdots & a_{k\,r-1} & a_{kr} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{k+1\,k} & \cdots & a_{k+1\,r-1} & a_{k+1\,r} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{r-1\,k} & \cdots & a_{r-1\,r-1} & a_{r-1\,r} \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{rk} & \cdots & a_{r\,r-1} & a_{rr}
\end{array} \right|$$

where the minor by which Δ is multiplied is enclosed. Further, observe that the $n-r$ rows of Δ not included in the chosen rows are prolonged in Δ' with the elements of these same rows repeated in order of the columns beginning with the kth. Now add the kth row of Δ' to the $(n+1)$th, the $(k+1)$th row to the $(n+2)$th, and so on, finally adding the rth row to the last. Afterward subtract the $(n+1)$th column of Δ' from the kth, the $(n+2)$th column from the $(k+1)$th, and so on, finally subtracting the last column from the rth. Then

$$\Delta' =$$

$$\left|\begin{array}{cccccccccccc}
a_{11} & a_{12} & \cdots a_{1\,k-1} & a_{1k} & \cdots a_{1\,r-1} & a_{1r} & a_{1\,r+1} & \cdots a_{1n} & 0 & \cdots & 0 & 0 \\
a_{21} & a_{22} & \cdots a_{2\,k-1} & a_{2k} & \cdots a_{2\,r-1} & a_{2r} & a_{2\,r+1} & \cdots a_{2n} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots\;\cdots & \cdots\;\cdots & \cdots & \cdots & \cdots & \cdots\;\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{k1} & a_{k2} & \cdots a_{k\,k-1} & a_{kk} & \cdots a_{k\,r-1} & a_{kr} & a_{k\,r+1} & \cdots a_{kn} & 0 & \cdots & 0 & 0 \\
a_{k+1\,1} & a_{k+1\,2} & \cdots a_{k+1\,k-1} & a_{k+1\,k} & \cdots a_{k+1\,r-1} & a_{k+1\,r} & a_{k+1\,r+1} & \cdots a_{k+1\,n} & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots\;\cdots & \cdots\;\cdots & \cdots & \cdots & \cdots & \cdots\;\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{r-1\,1} & a_{r-1\,2} & \cdots a_{r-1\,k-1} & a_{r-1\,k} & \cdots a_{r-1\,r-1} & a_{r-1\,r} & a_{r-1\,r+1} & \cdots a_{r-1\,n} & 0 & \cdots & 0 & 0 \\
a_{r1} & a_{r2} & \cdots a_{r\,k-1} & a_{rk} & \cdots a_{r\,r-1} & a_{rr} & a_{r\,r+1} & \cdots a_{rn} & 0 & \cdots & 0 & 0 \\
a_{r+1\,1} & a_{r+1\,2} & \cdots a_{r+1\,k-1} & 0 & \cdots\;\;0 & 0 & a_{r+1\,r+1} & \cdots a_{r+1\,n} & a_{r+1\,k} & \cdots & a_{r+1\,r-1} & a_{r+1\,r} \\
\cdots & \cdots & \cdots\;\cdots & \cdots\;\cdots & \cdots & \cdots & \cdots & \cdots\;\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n1} & a_{n2} & \cdots a_{n\,k-1} & 0 & \cdots\;\;0 & 0 & a_{n\,r+1} & \cdots a_{nn} & a_{nk} & \cdots & a_{n\,r-1} & a_{nr} \\
a_{k1} & a_{k2} & \cdots a_{k\,k-1} & 0 & \cdots\;\;0 & 0 & a_{k\,r+1} & \cdots a_{kn} & a_{kk} & \cdots & a_{k\,r-1} & a_{kr} \\
a_{k+1\,1} & a_{k+1\,2} & \cdots a_{k+1\,k-1} & 0 & \cdots\;\;0 & 0 & a_{k+1\,r+1} & \cdots a_{k+1\,n} & a_{k+1\,k} & \cdots & a_{k+1\,r-1} & a_{k+1\,r} \\
\cdots & \cdots & \cdots\;\cdots & \cdots\;\cdots & \cdots & \cdots & \cdots & \cdots\;\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{r-1\,1} & a_{r-1\,2} & \cdots a_{r-1\,k-1} & 0 & \cdots\;\;0 & 0 & a_{r-1\,r+1} & \cdots a_{r-1\,n} & a_{r-1\,k} & \cdots & a_{r-1\,r-1} & a_{r-1\,r} \\
a_{r1} & a_{r2} & \cdots a_{r\,k-1} & 0 & \cdots\;\;0 & 0 & a_{r\,r+1} & \cdots a_{rn} & a_{rk} & \cdots & a_{r\,r-1} & a_{rr}
\end{array}\right|$$

By **55** Δ' can be decomposed into products of pairs of minors, viz., the minors of the rth order formed from the first r rows and their complementaries. Since the elements in the columns of Δ' directly below M are zeros, all the minors of the rth order, formed from the first r rows, will have complementaries that vanish unless the said minors contain the given minor M. Hence the first factors of the products in the expansion of Δ' will all be minors of Δ, of the rth order, that contain the given

minor. Further, each complementary of such a factor is made up of the $n-r$ rows of Δ not found in the first factor and the $r-k+1$ rows in which M is found. Which proves the theorem.

$$|a_1\,b_2\,c_3\,d_4\,e_5|\,|b_2\,c_3| = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & e_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & e_3 & 0 & 0 \\ a_4 & b_4 & c_4 & d_4 & e_4 & b_4 & c_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 & b_5 & c_5 \\ 0 & 0 & 0 & 0 & 0 & b_2 & c_2 \\ 0 & 0 & 0 & 0 & 0 & b_3 & c_3 \end{vmatrix} \begin{aligned} &= |a_1\,b_2\,c_3|\,|d_4\,e_5\,b_2\,c_3| \\ &\quad - |b_1\,c_2\,d_3|\,|a_4\,e_5\,b_2\,c_3| \\ &\quad - |b_1\,c_2\,e_3|\,|a_4\,d_5\,b_2\,c_3|\,; \end{aligned}$$

$$|a_1 b_2 c_3 d_4 e_5|\,|b_2 c_3 d_4| = |a_1 b_2 c_3 d_4|\,|e_5 b_2 c_3 d_4| + |b_1 c_2 d_3 e_4|\,|a_5 b_2 c_3 d_4|.$$

The student may show (change the rows into columns before applying the theorem)

$$\begin{aligned} |a_1 b_2 c_3 d_4 e_5|\,|b_3 c_4| &= |a_1 b_2 c_3 d_4|\,|b_3 c_4 e_5| - |a_1 b_3 c_4 d_5|\,|b_3 c_4 e_2| \\ &\quad + |a_2 b_3 c_4 d_5|\,|b_3 c_4 e_1|\,; \end{aligned}$$

$$\begin{aligned} &|a_1 b_2 c_3 d_4 e_5|\,d_2 \\ &\quad = |c_1 d_2 e_3|\,|a_4 b_5 d_2| - |c_1 d_2 e_4|\,|a_3 b_5 d_2| + |c_1 d_2 e_5|\,|a_3 b_4 d_2| \\ &\quad + |c_2 d_3 e_4|\,|a_1 b_5 d_2| - |c_2 d_3 e_5|\,|a_1 b_4 d_2| + |c_2 d_4 e_5|\,|a_1 b_3 d_2|, \\ &= -|d_1 e_2|\,|a_3 b_4 c_5 d_2| + |d_2 e_3|\,|a_1 b_4 c_5 d_2| - |d_2 e_4|\,|a_1 b_3 c_5 d_2| \\ &\qquad + |d_2 e_5|\,|a_1 b_3 c_4 d_2|, \\ &= \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad . \end{aligned}$$

The second illustration given is especially interesting as it shows the form of the product when the minor is of order $n-2$. In that case the chosen rows are $n-1$ in number, and the development consists only of two terms, each term being the product of two determinants of the $(n-1)$th order. If we change the order of rows and columns in the result, we have

$$|a_1 b_2 c_3 d_4 e_5|\,|b_2 c_3 d_4| = |a_1 b_2 c_3 d_4|\,|b_2 c_3 d_4 e_5| - |b_1 c_2 d_3 e_4|\,|a_2 b_3 c_4 d_5|,$$

or
$$\Delta\ \Delta_{a_1,\,e_5} = \Delta_{e_5}\ \Delta_{a_1} - \Delta_{a_5}\ \Delta_{e_1}\,;$$

and, in general,

$$\Delta\ \Delta_{a_{ik},\,a_{pq}} = \Delta_{a_{ik}}\ \Delta_{a_{pq}} - \Delta_{a_{iq}}\ \Delta_{a_{pk}}.$$

Employing an obvious extension of the notation described in the latter part of **39**, the last formula becomes

$$\Delta\,\frac{d^2\Delta}{da_{ik}\,da_{pq}} = \frac{d\Delta}{da_{ik}}\,\frac{d\Delta}{da_{pq}} - \frac{d\Delta}{da_{iq}}\,\frac{d\Delta}{da_{pk}}.$$

Rectangular Arrays or Matrices.

58. As a determinant is a function of n^2 quantities, the elements are always found in a square array. It is often necessary to consider the determinant obtained by applying the process of **53** to two rectangular arrays of elements, *i.e.*, arrays in which the number of rows is not equal to the number of columns. We will now investigate the value of this product.

1st. When the number of columns exceeds the number of rows:

The product of two arrays (matrices) of elements in which the number of columns (m) exceeds the number of rows (n), is a determinant which is equal to the sum of all the products in which the first factor is a determinant of the nth order formed from the first array (matrix), and the second factor is the corresponding determinant of the nth order formed from the other array (matrix). Let the two arrays of elements be

$$\left.\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2n} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & \cdots & a_{nm} \end{matrix}\right\} \quad \text{and} \quad \left.\begin{matrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & \cdots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} & \cdots & \alpha_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} & \cdots & \alpha_{nm} \end{matrix}\right\}, \quad n < m.$$

Applying the process of **53**, we have the determinant

$$\Delta \equiv \begin{vmatrix} a_{11}\alpha_{11}+\cdots+a_{1n}\alpha_{1n}+\cdots+a_{1m}\alpha_{1m} & a_{11}\alpha_{21}+\cdots+a_{1n}\alpha_{2n}+\cdots+a_{1m}\alpha_{2m} & \cdots & a_{11}\alpha_{n1}+\cdots+a_{1n}\alpha_{nn}+\cdots+a_{1m}\alpha_{nm} \\ a_{21}\alpha_{11}+\cdots+a_{2n}\alpha_{1n}+\cdots+a_{2m}\alpha_{1m} & a_{21}\alpha_{21}+\cdots+a_{2n}\alpha_{2n}+\cdots+a_{2m}\alpha_{2m} & \cdots & a_{21}\alpha_{n1}+\cdots+a_{2n}\alpha_{nn}+\cdots+a_{2m}\alpha_{nm} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}\alpha_{11}+\cdots+a_{nn}\alpha_{1n}+\cdots+a_{nm}\alpha_{1m} & a_{n1}\alpha_{21}+\cdots+a_{nn}\alpha_{2n}+\cdots+a_{nm}\alpha_{2m} & \cdots & a_{n1}\alpha_{n1}+\cdots+a_{nn}\alpha_{nn}+\cdots+a_{nm}\alpha_{nm} \end{vmatrix}.$$

Now we may form from Δ a number of determinants $\Delta_1, \Delta_2, \Delta_3 \cdots$ of the nth order, the elements of which are all polynomials con-

sisting of n terms each. The number of such determinants is, of course, $\dfrac{m(m-1)(m-2)\cdots(m-n+1)}{n!}$. Let us consider one of these determinants; take, for example Δ_1, whose columns are formed from the first n terms in the columns of Δ. We have, accordingly,

$$\Delta_1 \equiv \begin{vmatrix} a_{11}\alpha_{11}+a_{12}\alpha_{12}+\cdots+a_{1n}\alpha_{1n} & a_{11}\alpha_{21}+a_{12}\alpha_{22}+\cdots+a_{1n}\alpha_{2n} & \cdots & a_{11}\alpha_{n1}+a_{12}\alpha_{n2}+\cdots+a_{1n}\alpha_{nn} \\ a_{21}\alpha_{11}+a_{22}\alpha_{12}+\cdots+a_{2n}\alpha_{1n} & a_{21}\alpha_{21}+a_{22}\alpha_{22}+\cdots+a_{2n}\alpha_{2n} & \cdots & a_{21}\alpha_{n1}+a_{22}\alpha_{n2}+\cdots+a_{2n}\alpha_{nn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}\alpha_{11}+a_{n2}\alpha_{12}+\cdots+a_{nn}\alpha_{1n} & a_{n1}\alpha_{21}+a_{n2}\alpha_{22}+\cdots+a_{nn}\alpha_{2n} & \cdots & a_{n1}\alpha_{n1}+a_{n2}\alpha_{n2}+\cdots+a_{nn}\alpha_{nn} \end{vmatrix}.$$

Now Δ_1 is, by **53**, the product of two factors, the first of which is the determinant formed from the first n columns of the first array of elements, and the second is the determinant formed from the corresponding n columns of the second array. In a similar manner we may show that each of the determinants $\Delta_1, \Delta_2, \Delta_3 \cdots$ is the product of two factors, each factor being a determinant formed from n corresponding columns of the two given arrays. Then in order to establish the proposition it remains to be shown that $\Delta = \Delta_1 + \Delta_2 + \Delta_3 + \cdots$. Each of the determinants $\Delta_1, \Delta_2, \Delta_3 \cdots$ can be decomposed into $n!$ non-vanishing determinants whose elements are monomials. Accordingly the sum $\Delta_1 + \Delta_2 + \Delta_3 + \cdots$ will contain

$$m(m-1)(m-2)\cdots(m-n+1)$$

non-vanishing determinants whose elements are monomials. Returning to Δ, we see that it can obviously be decomposed into m^n monomial element determinants; but those which do not vanish are only $m(m-1)(m-2)\cdots(m-n+1)$ in number. Now observing that each one of these monomial element determinants is a part of that one in the series $\Delta_1, \Delta_2, \Delta_3 \cdots$ in which its columns occur as parts of columns, the proposition is established.

Illustration:

Performing the operation of **53** upon

$$\left.\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \end{matrix}\right\} \text{ and } \left.\begin{matrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{matrix}\right\},$$

we obtain the determinant

$$\begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 + d_1\delta_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 + d_1\delta_2 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 + d_2\delta_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 + d_2\delta_2 \end{vmatrix}.$$

This determinant the student can readily show is equal to

$$(a_1b_2)(\alpha_1\beta_2) + (a_1c_2)(\alpha_1\gamma_2) + (a_1d_2)(\alpha_1\delta_2)$$
$$+ (b_1c_2)(\beta_1\gamma_2) + (b_1d_2)(\beta_1\delta_2) + (c_1d_2)(\gamma_1\delta_2).$$

2d. *When the number of rows exceeds the number of columns.*

Consider the two arrays.

$$\left.\begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}\right\} \text{ and } \left.\begin{matrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{matrix}\right\}.$$

Multiplying as before, we have

$$\Delta \equiv \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 & a_1\alpha_2 + b_1\beta_2 & a_1\alpha_3 + b_1\beta_3 \\ a_2\alpha_1 + b_2\beta_1 & a_2\alpha_2 + b_2\beta_2 & a_2\alpha_3 + b_2\beta_3 \\ a_3\alpha_1 + b_3\beta_1 & a_3\alpha_2 + b_3\beta_2 & a_3\alpha_3 + b_3\beta_3 \end{vmatrix} = 0.$$

The value of Δ is readily seen to be zero when we notice that it can be obtained by multiplying two determinants formed from the two given arrays by prefixing a column of zeros to each. The method of proof employed in this special case is general. It is only necessary to add to each array as many columns of zeros as are necessary to make each array square, and then compare the product of the two determinants thus formed with the determinant formed by compounding the two matrices.

*Reciprocal Determinants.**

59. If the principal minors of the elements of a determinant are themselves made the corresponding elements of another

* Reciprocal determinants would more properly be considered in the next chapter since they are among the "special forms," but for several reasons it is thought best to introduce them here.

determinant, the determinant thus formed is called the *reciprocal* or *adjugate determinant.* Or, in other words, the elements of the reciprocal determinant are the complementary minors of the corresponding elements in the original determinant.

The reciprocal of $(a_1\, b_2\, c_3)$ is

$$\begin{vmatrix} (b_2\, c_3) & -(a_2\, c_3) & (a_2\, b_3) \\ -(b_1\, c_3) & (a_1\, c_3) & -(a_1\, b_3) \\ (b_1\, c_2) & -(a_1\, c_2) & (a_1\, b_2) \end{vmatrix}.$$

Assimilating the notation of **19**, we have

$$|A_1\, B_2\, C_3 \ldots L_n|,\quad |A_{1n}|,\quad \text{or}\quad |A_{11}\, A_{22}\, A_{33} \ldots A_{nn}|,$$

for the determinant adjugate to

$$|\, a_1\;\; b_2\;\; c_3\; \ldots\; l_n\,|,\quad |\,a_{1n}\,|,\quad \text{or}\quad |\, a_{11}\;\; a_{22}\;\; a_{33}\; \ldots\; a_{nn}|,$$

respectively.

If the minus signs in the first illustration are erased, what is the effect upon the determinant? How is it in general?

60. Theorem. — *The determinant Δ' adjugate to any determinant Δ of the nth degree, equals the $(n-1)$th power of Δ.*

We have, for example,

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ and } \Delta' \equiv \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

Whence

$$\Delta\Delta' \equiv \begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix} = \Delta^3. \qquad \therefore\ \Delta' = \Delta^2.$$

The process here exemplified is perfectly general, hence the proposition.

61. Theorem. — *Any minor of the kth degree of the reciprocal determinant Δ' is equal to the complementary of the corresponding minor in the original determinant Δ multiplied by the $(k-1)$th power of Δ.*

Let $\Delta \equiv |\, a_{14}\,|$, and $\Delta' \equiv |\, A_{14}\,|$.

Transform Δ and Δ' so that the minors $|a_{11}\ a_{32}\ a_{44}|$ and $|A_{11}\ A_{32}\ A_{44}|$ occupy the first three rows and columns in their respective determinants. Then

$$\Delta = (-1)^{\mu} \begin{vmatrix} a_{11} & a_{12} & a_{14} & a_{13} \\ a_{31} & a_{32} & a_{34} & a_{33} \\ a_{41} & a_{42} & a_{44} & a_{43} \\ a_{21} & a_{22} & a_{24} & a_{23} \end{vmatrix},$$

$$\text{and } \Delta' = (-1)^{\mu} \begin{vmatrix} A_{11} & A_{12} & A_{14} & A_{13} \\ A_{31} & A_{32} & A_{34} & A_{33} \\ A_{41} & A_{42} & A_{44} & A_{43} \\ A_{21} & A_{22} & A_{24} & A_{23} \end{vmatrix}.$$

Then

$$|A_{11}\ A_{32}\ A_{44}| = (-1)^{\mu} \begin{vmatrix} A_{11} & A_{12} & A_{14} & A_{13} \\ A_{31} & A_{32} & A_{34} & A_{33} \\ A_{41} & A_{42} & A_{44} & A_{43} \\ 0 & 0 & 0 & 1 \end{vmatrix}.$$

Multiplying,

$$\Delta |A_{11}\ A_{32}\ A_{44}| = \begin{vmatrix} \Delta & 0 & 0 & a_{13} \\ 0 & \Delta & 0 & a_{33} \\ 0 & 0 & \Delta & a_{43} \\ 0 & 0 & 0 & a_{23} \end{vmatrix} = a_{23}\,\Delta^3.$$

Whence $|A_{11}\,A_{32}\,A_{44}| = a_{23}\,\Delta^2$, which is the required value of a first minor of Δ'.

To find the value of a second minor of Δ' we may proceed as follows:

The minor

$$|A_{22}\ A_{33}| = (-1)^{\mu} \begin{vmatrix} A_{22} & A_{23} & A_{21} & A_{24} \\ A_{32} & A_{33} & A_{31} & A_{34} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

and the corresponding form of Δ is

$$(-1)^{\mu} \begin{vmatrix} a_{22} & a_{23} & a_{21} & a_{24} \\ a_{32} & a_{33} & a_{31} & a_{34} \\ a_{12} & a_{13} & a_{11} & a_{14} \\ a_{42} & a_{43} & a_{41} & a_{44} \end{vmatrix}.$$

As before,

$$\Delta|A_{22}\ A_{33}| = \begin{vmatrix} \Delta & 0 & a_{21} & a_{24} \\ 0 & \Delta & a_{31} & a_{34} \\ 0 & 0 & a_{11} & a_{14} \\ 0 & 0 & a_{41} & a_{44} \end{vmatrix} = \Delta^2|a_{11}\ a_{44}|.$$

Whence, $|A_{22}\ A_{33}| = |a_{11}\ a_{44}|\Delta.$

The student may put

$$\Delta \equiv |a_1\ b_2\ c_3\ d_4| \text{ and } \Delta' = |A_1\ B_2\ C_3\ D_4|,$$

and then show that

$$|B_2\ C_3\ D_4| = a_1\Delta^2;$$
$$|A_1\ D_4| = |b_2\ c_3|\Delta.$$

The general theorem, of which the preceding are special cases, is proved as follows:

Let $\Delta \equiv |a_{1n}|$ and $\Delta' \equiv |A_{1n}|$,

and let the minor of the kth order of Δ' whose value is sought be

$$\Delta_k^* \equiv \begin{vmatrix} A_{p_1q_1} & A_{p_1q_2} & A_{p_1q_3} & \cdots & A_{p_1q_k} \\ A_{p_2q_1} & A_{p_2q_2} & A_{p_2q_3} & \cdots & A_{p_2q_k} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{p_kq_1} & A_{p_kq_2} & A_{p_kq_3} & \cdots & A_{p_kq_k} \end{vmatrix}.$$

Now putting

$$\mu \equiv p_1 + p_2 + \cdots + p_k + q_1 + q_2 + \cdots + q_k,$$

we may write

$$\Delta = (-1)^\mu$$

$$\begin{vmatrix} a_{p_1q_1} & \cdots & a_{p_1q_k} & a_{p_11} & \cdots & a_{p_1q_1-1} & a_{p_1q_1+1} & \cdots & a_{p_1q_2-1} & a_{p_1q_2+1} & \cdots & a_{p_1n} \\ a_{p_2q_1} & \cdots & a_{p_2q_k} & a_{p_21} & \cdots & a_{p_2q_1-1} & a_{p_2q_1+1} & \cdots & a_{p_2q_2-1} & a_{p_2q_2+1} & \cdots & a_{p_2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{p_kq_1} & \cdots & a_{p_kq_k} & a_{p_k1} & \cdots & a_{p_kq_1-1} & a_{p_kq_1+1} & \cdots & a_{p_kq_2-1} & a_{p_kq_2+1} & \cdots & a_{p_kn} \\ a_{1q_1} & \cdots & a_{1q_k} & a_{11} & \cdots & a_{1q_1-1} & a_{1q_1+1} & \cdots & a_{1q_2-1} & a_{1q_2+1} & \cdots & a_{1n} \\ a_{2q_1} & \cdots & a_{2q_k} & a_{21} & \cdots & a_{2q_1-1} & a_{2q_1+1} & \cdots & a_{2q_2-1} & a_{2q_2+1} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{nq_1} & \cdots & a_{nq_k} & a_{n1} & \cdots & a_{nq_1-1} & a_{nq_1+1} & \cdots & a_{nq_2-1} & a_{nq_2+1} & \cdots & a_{nn} \end{vmatrix}.$$

* In this determinant the subscripts $p_1, p_2, p_3, \ldots q_1, q_2, q_3, \ldots$ of course stand for any integers in order of magnitude.

The corresponding form of Δ_k is

$$\Delta_k = (-1)^\mu$$

$$\begin{vmatrix}
A_{p_1q_1} & \cdots & A_{p_1q_k} & A_{p_11} & \cdots & A_{p_1q_1-1} & A_{p_1q_1+1} & \cdots & A_{p_1q_2-1} & A_{p_1q_2+1} & \cdots & A_{p_1n} \\
A_{p_2q_1} & \cdots & A_{p_2q_k} & A_{p_21} & \cdots & A_{p_2q_1-1} & A_{p_2q_1+1} & \cdots & A_{p_2q_2-1} & A_{p_2q_2+1} & \cdots & A_{p_2n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A_{p_kq_1} & \cdots & A_{p_kq_k} & A_{p_k1} & \cdots & A_{p_kq_1-1} & A_{p_kq_1+1} & \cdots & A_{p_kq_2-1} & A_{p_kq_2+1} & \cdots & A_{p_kn} \\
0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 1_{(n-k)}
\end{vmatrix}.$$

We notice that this form of Δ_k is just the same as if it had been derived from Δ' by making the p_1th, p_2th, $\cdots p_k$th rows of Δ' the 1st, 2d, $\cdots k$th rows, making the same changes in the places of the q_1th, q_2th, ... q_kth columns, and then putting 1 for each remaining element of the principal diagonal, and 0 for every other element of the $n-k$ rows of which Δ_k is not a part.

Multiplying, we have

$$\Delta\Delta_k$$

$$=\begin{vmatrix}
\Delta & 0 & \cdots & 0 & a_{p_11} & \cdots & a_{p_1q_1-1} & a_{p_1q_1+1} & \cdots & a_{p_1q_2-1} & a_{p_1q_2+1} & \cdots & a_{p_1n} \\
0 & \Delta & \cdots & 0 & a_{p_21} & \cdots & a_{p_2q_1-1} & a_{p_2q_1+1} & \cdots & a_{p_2q_2-1} & a_{p_2q_2+1} & \cdots & a_{p_2n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \Delta & a_{p_k1} & \cdots & a_{p_kq_1-1} & a_{p_kq_1+1} & \cdots & a_{p_kq_2-1} & a_{p_kq_2+1} & \cdots & a_{p_kn} \\
0 & 0 & \cdots & 0 & a_{11} & \cdots & a_{1q_1-1} & a_{1q_1+1} & \cdots & a_{1q_2-1} & a_{1q_2+1} & \cdots & a_{1n} \\
0 & 0 & \cdots & 0 & a_{21} & \cdots & a_{2q_1-1} & a_{2q_1+1} & \cdots & a_{2q_2-1} & a_{2q_2+1} & \cdots & a_{2n} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 0 & a_{n1} & \cdots & a_{nq_1-1} & a_{nq_1+1} & \cdots & a_{nq_2-1} & a_{nq_2+1} & \cdots & a_{nn}
\end{vmatrix}.$$

Now this determinant is at once expressible as the product of two determinate factors, and we have

$\Delta\Delta_k = \Delta^k$ *times the complementary of the minor of* Δ *corresponding to* Δ_k *in* Δ'.

Whence

$\Delta_k = \Delta^{k-1}$ *times the complementary of the minor of* Δ *corresponding to* Δ_k *of* Δ',

as was to be shown.

62. From the preceding article it follows at once that if $\Delta = 0$, then

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = \begin{vmatrix} A_{31} & A_{3n} \\ A_{n1} & A_{nn} \end{vmatrix} = \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} = \cdots = 0;$$

i.e., in general

$$\begin{vmatrix} A_{ik} & A_{ie} \\ A_{pk} & A_{pe} \end{vmatrix} = 0,$$

whence

$$A_{ik} \quad A_{pe} = A_{pk} \quad A_{ie}$$

or

$$A_{ik} : A_{ie} :: A_{pk} : A_{pe}.$$

That is to say:

If $\Delta = 0$, *the cofactors of the elements of any row are proportional to the cofactors of the corresponding elements of any other row.*

From the preceding article we have also

$$\begin{vmatrix} A_{ik} & A_{ie} \\ A_{pk} & A_{pe} \end{vmatrix} = \Delta \times \textit{complementary minor of} \begin{vmatrix} a_{ik} & a_{ie} \\ a_{pk} & a_{pe} \end{vmatrix},$$

which may be written

$$\begin{vmatrix} \dfrac{d\Delta}{da_{ik}} & \dfrac{d\Delta}{da_{ie}} \\[2ex] \dfrac{d\Delta}{da_{pk}} & \dfrac{d\Delta}{da_{pe}} \end{vmatrix} = \Delta \frac{d^2\Delta}{da_{ik}da_{pe}};$$

whence

$$\Delta \frac{d^2\Delta}{da_{ik}da_{pe}} = \frac{d\Delta}{da_{ik}} \frac{d\Delta}{da_{pe}} - \frac{d\Delta}{da_{pk}} \frac{d\Delta}{da_{ie}},$$

which is the formula already obtained in **57**.

EXAMPLES.

1. Show that

$$\begin{vmatrix} z_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & y_2 & 0 \\ 0 & 0 & z_3 & y_3 & 0 & x_3 \\ 0 & z_3 & z_2 & y_2 & y_3 & x_2 \\ z_3 & 0 & z_1 & y_1 & 0 & x_1 \\ z_2 & z_1 & 0 & 0 & y_1 & 0 \end{vmatrix} = -z_1 |x_1 y_2 z_3| \;\; |y_1 z_2|;$$

also

$$\begin{vmatrix} a_1 & 0 & 0 & a_4 & 0 & 0 \\ 0 & a_2 & 0 & 0 & a_5 & 0 \\ 0 & 0 & a_3 & 0 & 0 & a_6 \\ b_1 & 0 & 0 & b_4 & 0 & 0 \\ 0 & b_2 & 0 & 0 & b_5 & 0 \\ 0 & 0 & b_3 & 0 & 0 & b_6 \end{vmatrix} = |a_1 b_4| \;\; |a_2 b_5| \;\; |a_3 b_6|.$$

2. Show that

$$\begin{vmatrix} a_2 & b_2 & c_2 & 0 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 \\ a_1A_1 & b_1A_1 & c_1A_1 & A_2 & A_3 \\ a_1B_1 & b_1B_1 & c_1B_1 & B_2 & B_3 \\ a_1C_1 & b_1C_1 & c_1C_1 & C_2 & C_3 \end{vmatrix} = |A_1 B_2 C_3| \;\; |a_1 b_2 c_3|.$$

3. If Δ is a determinant of the nth order, having $n-m$ zero elements in the corresponding places of m rows, then Δ is the product of that minor whose elements are the other elements of the m rows and its complementary; the sign of the product is determined as in **55**.

4. If any determinant of the nth order has more than $(n-m)$ zero elements in the corresponding places of m rows, the determinant vanishes.

5. $$\begin{vmatrix} a_1 & b_1 & c_1 & M & N \\ a_2 & b_2 & c_2 & P & Q \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ a_4 & b_4 & c_4 & a_4 & b_4 \\ a_5 & b_5 & c_5 & a_5 & b_5 \end{vmatrix} = \begin{vmatrix} a_1 - M & b_1 - N \\ a_2 - P & b_2 - Q \end{vmatrix} \;\; |c_3 a_4 b_5|;$$

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & 0 & 0 & b_4 \\ c_1 & 0 & 0 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = |d_2 a_3| \;\; |b_4 c_1|.$$

6.
$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} \\ 0 & 0 & 0 & 0 & 0 & c_6 & c_7 & c_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_6 & d_7 & d_8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_6 & e_7 & e_8 & 0 & 0 \\ f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 & 0 & 0 \\ g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 & g_8 & 0 & 0 \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & 0 & 0 \\ i_1 & i_2 & 0 & 0 & 0 & i_6 & i_7 & i_8 & 0 & 0 \\ k_1 & k_2 & 0 & 0 & 0 & k_6 & k_7 & k_8 & 0 & 0 \end{vmatrix}$$
$$= -|a_9 b_{10}|\ |c_6 d_7 e_8|\ |f_3 g_4 h_5|\ |i_1 k_2|.$$

7. Show that
$$\begin{vmatrix} (a_1-b_1)^2 & (a_1-b_2)^2 & (a_1-b_3)^2 & (a_1-b_4)^2 \\ (a_2-b_1)^2 & (a_2-b_2)^2 & (a_2-b_3)^2 & (a_2-b_4)^2 \\ (a_3-b_1)^2 & (a_3-b_2)^2 & (a_3-b_3)^2 & (a_3-b_4)^2 \\ (a_4-b_1)^2 & (a_4-b_2)^2 & (a_4-b_3)^2 & (a_4-b_4)^2 \end{vmatrix} = 0.$$

This may be proved by multiplying the two arrays:
$$\left.\begin{matrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \\ a_4^2 & a_4 & 1 \end{matrix}\right\} \text{ and } \left.\begin{matrix} 1 & -2b_1 & b_1^2 \\ 1 & -2b_2 & b_2^2 \\ 1 & -2b_3 & b_3^2 \\ 1 & -2b_4 & b_4^2 \end{matrix}\right\}.$$

8. Show that
$$|a_{1n}|(x_1 + x_2 + x_3 + \cdots + x_n)$$
$$= \begin{vmatrix} a_{11}x_1 & a_{12}x_2 & \cdots & a_{1n}x_n \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21}x_1 & a_{22}x_2 & \cdots & a_{2n}x_n \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$
$$+ \cdots + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}x_1 & a_{n2}x_2 & \cdots & a_{nn}x_n \end{vmatrix}.$$

Notice that the coefficient of x_i in this sum is
$$a_{1i}A_{1i} + a_{2i}A_{2i} + a_{3i}A_{3i} + \cdots + a_{ni}A_{ni} = |a_{1n}|.$$

9. As an application of the preceding, show that
$$2(x_1 + x_2 + x_3)\begin{vmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & x_1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_2 & x_3 & x_1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ x_2^2 & x_3^2 & x_1^2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} x_1x_2 & x_2x_3 & x_3x_1 \\ x_2 & x_3 & x_1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} x_1 & x_2 & x_3 \\ x_1x_2 & x_2x_3 & x_3x_1 \\ 1 & 1 & 1 \end{vmatrix}.$$

10. Given $f_1(x) = a_1x^3 + 3\,b_1x^2 + 3\,c_1x + d_1,$

$f_2(x) = a_2x^3 + 3\,b_2x^2 + 3\,c_2x + d_2,$

$f_3(x) = a_3x^3 + 3\,b_3x^2 + 3\,c_3x + d_3;$

show that

$$\begin{vmatrix} f_1(x) & f_1'(x) & f_1''(x) \\ f_2(x) & f_2'(x) & f_2''(x) \\ f_3(x) & f_3'(x) & f_3''(x) \end{vmatrix} \equiv -18 \begin{vmatrix} 1 & -x & x^2 & -x^3 \\ a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}.$$

The first determinant is at once reducible to

$$-18 \begin{vmatrix} a_1x + b_1 & b_1x + c_1 & c_1x + d_1 \\ a_2x + b_2 & b_2x + c_2 & c_2x + d_2 \\ a_3x + b_3 & b_3x + c_3 & c_3x + d_3 \end{vmatrix},$$

which may be written

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ a_1 & a_1x + b_1 & b_1x + c_1 & c_1x + d_1 \\ a_2 & a_2x + b_2 & b_2x + c_2 & c_2x + d_2 \\ a_3 & a_3x + b_3 & b_3x + c_3 & c_3x + d_3 \end{vmatrix}.$$

Again using **37**, the last determinant becomes the result above written. The student's attention is called to the fact that the method of *bordering* a determinant, *i.e.*, increasing its degree without changing its value, here employed, is frequently of use in simplifying.

63. The following examples comprise several interesting expansions of determinants. The cases considered and the methods employed are important.

I. Expand the following determinant in ascending powers of x:

$$\Delta \equiv \begin{vmatrix} a_{11} + x & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} + x & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} + x \end{vmatrix}.$$

Δ is evidently a function of x of the nth degree, in which the coefficient of x^n is 1, and the absolute term is $f(0) \equiv |a_{1n}|$. To complete the expansion, we have to find the coefficient of x^k.

Consider the product of two complementary minors of Δ, of the kth and $(n-k)$th degrees respectively,

$$\begin{vmatrix} a_{ee}+x & a_{eg} & \cdots \\ a_{ge} & a_{gg}+x & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} \text{ and } \begin{vmatrix} a_{pp}+x & a_{pq} & \cdots \\ a_{qp} & a_{qq}+x & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix}.$$

This product contains the term

$$x^k \begin{vmatrix} a_{pp} & a_{pq} & \cdots & a_{pn} \\ a_{qp} & a_{qq} & \cdots & a_{qn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{np} & a_{nq} & \cdots & a_{nn} \end{vmatrix} \equiv x^k D_{n-k}, \text{ say.}$$

The entire coefficient of x^k is accordingly ΣD_{n-k}, *i.e.*, the sum of all the minors of $|a_{1n}|$ of order $n-k$, whose principal diagonal lies in the principal diagonal of $|a_{1n}|$.

$$\therefore\ \Delta = |a_{1n}| + x\Sigma D_{n-1} + x^2 \Sigma D_{n-2} + \cdots + x^n.$$

As an illustration, the student may show that

$$\begin{vmatrix} a_1+x & b_1 & c_1 & d_1 \\ a_2 & b_2+x & c_2 & d_2 \\ a_3 & b_3 & c_3+x & d_3 \\ a_4 & b_4 & c_4 & d_4+x \end{vmatrix}$$

$$\begin{aligned} = |a_1 b_2 c_3 d_4| + [\,|b_2 c_3 d_4| &+ |a_1 c_3 d_4| \\ &+ |a_1 b_2 d_4| + |a_1 b_2 c_3|\,]\,x \\ &+ [\,|b_2 c_3| + |a_1 d_4| + |a_1 c_3| \\ &+ |b_2 d_4| + |a_1 b_2| + |c_3 d_4|\,]\,x^2 \\ &+ [a_1 + b_2 + c_3 + d_4]\,x^3 + x^4. \end{aligned}$$

For another exercise, let the student find the terms of $\Delta \equiv |a_{1n}|$ that contain k elements from the principal diagonal, by considering the product of two complementary minors, as above.

II. Expand

$$\Delta \equiv \begin{vmatrix} ax+ly & c_1x+n_1y & b_1x+m_1y \\ c_2x+n_2y & bx+my & a_1x+l_1y \\ b_2x+m_2y & a_2x+l_2y & cx+ny \end{vmatrix}$$

in ascending powers of x and y.

Putting first $y=0$, and then $x=0$, the terms involving x^3 and y^3, respectively, are

$$x^3\begin{vmatrix} a & c_1 & b_1 \\ c_2 & b & a_1 \\ b_2 & a_2 & c \end{vmatrix}, \text{ and } y^3\begin{vmatrix} l & n_1 & m_1 \\ n_2 & m & l_1 \\ m_2 & l_2 & n \end{vmatrix}.$$

Putting the y's in the two last columns of Δ equal to zero, we obtain for one set of terms involving x^2y

$$x^2y\begin{vmatrix} l & c_1 & b_1 \\ n_2 & b & a_1 \\ m_2 & a_2 & c \end{vmatrix},$$

and the two other sets of terms containing x^2y are, similarly,

$$x^2y\begin{vmatrix} a & n_1 & b_1 \\ c_2 & m & a_1 \\ b_2 & l_2 & c \end{vmatrix} \text{ and } x^2y\begin{vmatrix} a & c_1 & m_1 \\ c_2 & b & l_1 \\ b_2 & a_2 & n \end{vmatrix}.$$

The coefficient of xy^2 is found in a similar manner, and the entire expansion is accordingly

$$\Delta = x^3\begin{vmatrix} a & c_1 & b_1 \\ c_2 & b & a_1 \\ b_2 & a_2 & c \end{vmatrix} + x^2y\left[\begin{vmatrix} l & c_1 & b_1 \\ n_2 & b & a_1 \\ m_2 & a_2 & c \end{vmatrix} + \begin{vmatrix} a & n_1 & b_1 \\ c_2 & m & a_1 \\ b_2 & l_2 & c \end{vmatrix} + \begin{vmatrix} a & c_1 & m_1 \\ c_2 & b & l_1 \\ b_2 & a_2 & n \end{vmatrix}\right]$$

$$+ xy^2\left[\begin{vmatrix} a & n_1 & m_1 \\ c_2 & m & l_1 \\ b_2 & l_2 & n \end{vmatrix} + \begin{vmatrix} l & c_1 & m_1 \\ n_2 & b & l_1 \\ m_2 & a_2 & n \end{vmatrix} + \begin{vmatrix} l & n_1 & b_1 \\ n_2 & m & a_1 \\ m_2 & l_2 & c \end{vmatrix}\right] + y^3\begin{vmatrix} l & n_1 & m_1 \\ n_2 & m & l_1 \\ m_2 & l_2 & n \end{vmatrix}.$$

III. Show that any determinant Δ may be developed in terms of the elements of any row and column and the second minors of Δ corresponding to the product of these elements.

Let $$\Delta' \equiv |a_{11}\ a_{22}\ a_{33}|,$$

and border it as indicated below; calling the result Δ, we may

expand Δ in terms of the bordering elements and first minors of Δ', *i.e.*,

$$\Delta \equiv \begin{vmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{aligned} a_{00}\ \Delta' - \{ & a_{10}\ a_{01}\ A_{11} + a_{10}\ a_{02}\ A_{12} \\ & + a_{10}\ a_{03}\ A_{13} + a_{20}\ a_{01}\ A_{21} + a_{20}\ a_{02}\ A_{22} \\ & + a_{20}\ a_{03}\ A_{23} + a_{30}\ a_{01}\ A_{31} + a_{30}\ a_{02}\ A_{32} \\ & + a_{30}\ a_{03}\ A_{33} \}, \end{aligned}$$

in which A_{ik} is, as usual, a first minor (with its proper sign) of Δ'.

In general, if $\Delta' \equiv | a_{11}\, a_{22} \cdots a_{nn} |$, we have

$$\Delta \equiv \begin{vmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0n} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1n} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n0} & a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{00}\Delta' - \Sigma\, a_{i0}\, a_{0k} A_{ik} \quad (i, k = 1, 2, 3 \ldots n),$$

?

in which, as before, A_{ik} is a minor of Δ'.

For the terms of Δ containing a_{00} are obviously $a_{00}\Delta'$. Now let C be the complementary minor of

$$\begin{vmatrix} a_{00} & a_{0k} \\ a_{i0} & a_{ik} \end{vmatrix} \text{ in } \Delta;$$

then $a_{00} a_{ik} C$ contains all the terms of Δ involving $a_{00} a_{ik}$; hence $a_{ik} C$ contains all the terms of Δ' involving a_{ik}, and consequently

$$C = A_{ik},$$

and $-\, a_{i0}\, a_{0k} A_{ik}$ is the expression for the terms of Δ containing the bordering elements a_{i0}, a_{0k}.

This expansion, known as Cauchy's Theorem, is frequently written

$$\Delta = a_{rs}\, A_{rs} - \Sigma\, a_{rk}\, a_{is}\, \beta_{ik}. \qquad (a)$$

Here Δ is a determinant of the nth order. A_{rs} is, as usual, the complementary minor of a_{rs} in Δ; i has all integral values from 1 to n, except r; k has all integral values from 1 to n, except s; and β_{ik} is the complementary minor of a_{ik} in A_{rs}. (a) is, accordingly, the expansion of Δ in terms of the elements of the rth row and the sth column.

The student may show that

$$\Delta \equiv \begin{vmatrix} a & f & g & h \\ f_1 & b & 0 & 0 \\ g_1 & 0 & c & 0 \\ h_1 & 0 & 0 & d \end{vmatrix} = abcd - ff_1cd - gg_1bd - hh_1bc.$$

$$\Delta \equiv \begin{vmatrix} a_1+x_1 & a_2 & a_3 & a_4 \\ -x_1 & x_2 & 0 & 0 \\ 0 & -x_2 & x_3 & 0 \\ 0 & 0 & -x_3 & x_4 \end{vmatrix} = \begin{vmatrix} a_1+x_1 & a_2 & a_3 & a_4 \\ -x_1 & x_2 & 0 & 0 \\ -x_1 & 0 & x_3 & 0 \\ -x_1 & 0 & 0 & x_4 \end{vmatrix}$$

$$= x_1 x_2 x_3 x_4 \left\{ 1 + \frac{a_1}{x_1} + \frac{a_2}{x_2} + \frac{a_3}{x_3} + \frac{a_4}{x_4} \right\}.$$

IV. If $\Delta \equiv \begin{vmatrix} x_1 & a_2 & a_3 & \dots & a_n \\ a_1 & x_2 & a_3 & \dots & a_n \\ a_1 & a_2 & x_3 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x_n \end{vmatrix}$;

and if we put

$$f(x) \equiv (x_1 - a_1)\,(x_2 - a_2) \cdots (x_n - a_n),$$

and

$$f'(x_i) \equiv \frac{df(x)}{dx_i} = (x_1 - a_1) \cdots (x_{i-1} - a_{i-1})\,(x_{i+1} a_{i+1}) \cdots (x_n - a_n),$$

we find $$\Delta = f(x) + \Sigma a_i f'(x_i).$$

For

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & x_1 & a_2 & a_3 & \dots & a_n \\ 1 & a_1 & x_2 & a_3 & \dots & a_n \\ 1 & a_1 & a_2 & x_3 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & a_1 & a_2 & a_3 & \dots & x_n \end{vmatrix} = \begin{vmatrix} 1 & -a_1 & -a_2 & -a_3 & \dots & -a_n \\ 1 & x_1 - a_1 & 0 & 0 & \dots & 0 \\ 1 & 0 & x_2 - a_2 & 0 & \dots & 0 \\ 1 & 0 & 0 & x_3 - a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots & x_n - a_n \end{vmatrix};$$

whence (if, as in III., we let Δ' represent the complementary minor of the first element) $\Delta' = f(x)$, and, since every first minor of Δ' vanishes except the minors of the diagonal elements, we have the required value of Δ on applying the theorem

$$\Delta = a_{00}\Delta' - \Sigma\, a_{i0}\, a_{0k}\, A_{ik}.$$

V. Show that

$$\Delta \equiv \begin{vmatrix} a_1 & x & x & x & \dots & x \\ x & a_2 & x & x & \dots & x \\ x & x & a_3 & x & \dots & x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ x & x & x & x & \dots & a_n \end{vmatrix} = f(x) - xf'(x),$$

in which $f(x) \equiv (x - a_1)(x - a_2)(x - a_3) \cdots (x - a_n)$,

and
$$f'(x) \equiv \frac{df(x)}{dx} = (x - a_2)(x - a_3) \cdots (x - a_n)$$
$$+ (x - a_1)(x - a_3) \cdots (x - a_n)$$
$$+ \cdots + (x - a_1)(x - a_2) \cdots (x - a_{n-1}).$$

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & a_1 & x & x & \dots & x \\ 1 & x & a_2 & x & \dots & x \\ 1 & x & x & a_3 & \dots & x \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x & x & x & \dots & a_n \end{vmatrix} = \begin{vmatrix} 1 & -x & -x & -x & \dots & -x \\ 1 & a_1 - x & 0 & 0 & \dots & 0 \\ 1 & 0 & a_2 - x & 0 & \dots & 0 \\ 1 & 0 & 0 & a_3 - x & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots & a_n - x \end{vmatrix}.$$

Then, as in the preceding example,

$$\Delta = f(x) - xf'(x).$$

64. To the expansions of the preceding article we append the solutions of the following determinant equations.

I. Solve the equation

$$\Delta \equiv \begin{vmatrix} x & a_1 & a_1 & a_1 \\ a_1 & x & a_1 & a_1 \\ a_1 & a_1 & x & a_1 \\ a_1 & a_1 & a_1 & x \end{vmatrix} = 0.$$

We find by easy reductions

$$\Delta = (x - a_1)^3 \begin{vmatrix} x & a_1 & a_1 & a_1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix} = (x - a_1)^3 (x + 3a_1) = 0.$$

Whence, $x = a_1, \ a_1, \ a_1, \ -3a_1.$

II. Find the values of x in the equation

$$\Delta \equiv \begin{vmatrix} x & a_1 & b_1 & c_1 \\ a_1 & x & c_1 & b_1 \\ b_1 & c_1 & x & a_1 \\ c_1 & b_1 & a_1 & x \end{vmatrix} = 0.$$

$$\Delta = \begin{vmatrix} x+a_1+b_1+c_1 & a_1 & b_1 & c_1 \\ x+a_1+b_1+c_1 & x & c_1 & b_1 \\ x+a_1+b_1+c_1 & c_1 & x & a_1 \\ x+a_1+b_1+c_1 & b_1 & a_1 & x \end{vmatrix} = (x+a_1+b_1+c_1) \begin{vmatrix} 1 & a_1 & b_1 & c_1 \\ 1 & x & c_1 & b_1 \\ 1 & c_1 & x & a_1 \\ 1 & b_1 & a_1 & x \end{vmatrix}$$

$$= (x+a_1+b_1+c_1)(x-a_1+b_1-c_1) \begin{vmatrix} 0 & -1 & 1 & -1 \\ 1 & x & c_1 & b_1 \\ 1 & c_1 & x & a_1 \\ 1 & b_1 & a_1 & x \end{vmatrix}.$$

Put the two polynomial factors $\equiv A$ and B respectively; then the last expression

$$= A.\,B. \begin{vmatrix} 0 & 0 & -1 & 0 \\ 1 & x+c_1 & c_1 & b_1+c_1 \\ 1 & x+c_1 & x & a_1+x \\ 1 & b_1+a_1 & a_1 & a_1+x \end{vmatrix}$$

$$= A.\,B. \begin{vmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & c_1 & b_1+c_1-a_1-x \\ 1 & 0 & x & 0 \\ 1 & b_1+a_1-x-c_1 & a_1 & 0 \end{vmatrix}.$$

Whence

$$(x+a_1+b_1+c_1)(x-a_1-c_1+b_1)(b_1+a_1-x-c_1)$$
$$(a_1+x-b_1-c_1) = 0.$$

$$\therefore x = -(a_1+b_1+c_1),\ (a_1-b_1+c_1),\ (b_1-c_1+a_1),$$
$$(b_1-a_1+c_1).$$

III. Find the roots of the equation

$$\Delta \equiv \begin{vmatrix} a^3 & b^3 & c^3 \\ (a+\lambda)^3 & (b+\lambda)^3 & (c+\lambda)^3 \\ (2a+\lambda)^3 & (2b+\lambda)^3 & (2c+\lambda)^3 \end{vmatrix} = 0.$$

From the third row of Δ subtract the first row multiplied by 8,

and from the second row subtract the first row. Then subtract the second row from the third, and we have

$$\Delta = 3\lambda^2 \begin{vmatrix} a^3 & b^3 & c^3 \\ 3a^2+3a\lambda+\lambda^2 & 3b^2+3b\lambda+\lambda^2 & 3c^2+3c\lambda+\lambda_2 \\ 3a^2+a\lambda & 3b^2+b\lambda & 3c^2+c\lambda \end{vmatrix}.$$

Now subtract the third row from the second, and

$$\Delta = 3\lambda^3 \begin{vmatrix} a^3 & b^3 & c^3 \\ 2a+\lambda & 2b+\lambda & 2c+\lambda \\ 3a^2+a\lambda & 3b^2+b\lambda & 3c^2+c\lambda \end{vmatrix} = 0.$$

From this equation it is obvious that three values of λ are zero; the other two roots can be found by equating to zero the quadratic factor of the first number, and solving for λ.

Δ may, however, be further simplified as follows: subtract the first column from each of the other two; then

$$\Delta = 3\lambda^3(c-a)(b-a) \begin{vmatrix} a^3 & b^2+ab+a^2 & c^2+ac+a^2 \\ 2a+\lambda & 2 & 2 \\ 3a^2+a\lambda & 3b+3a+\lambda & 3c+3a+\lambda \end{vmatrix}.$$

Now subtract the second column from the third, and

$$\Delta = 3\lambda^3(b-a)(c-a)(c-b) \begin{vmatrix} a^3 & a^2+ab+b^2 & a+b+c \\ 2a+\lambda & 2 & 0 \\ 3a^2+a\lambda & 3a+3b+\lambda & 3 \end{vmatrix}.$$

Finally, add the second column multiplied by $-a$ and the third multiplied by ab to the first, and afterward subtract the third multiplied by $a+b$ from the second; then

$$\Delta = 3\lambda^3(b-a)(c-a)(c-b) \begin{vmatrix} abc & -bc-ca-ab & a+b+c \\ \lambda & 2 & 0 \\ 0 & \lambda & 3 \end{vmatrix} = 0.$$

Whence three values of λ are seen to be zero, and the other two roots are readily found from the quadratic

$$(a+b+c)\lambda^2 + 3(bc+ac+ab)\lambda + 6abc = 0.$$

65. THEOREM. — *The total differential of a determinant Δ is a sum of n determinants, each of which is obtained from Δ by substituting the differentials of the elements of a row for the elements themselves.*

Let $$\Delta \equiv | x_1 y_2 z_3 \cdots t_n |.$$

Developing in terms of the elements of the ith row,

$$\Delta = x_i X_i + y_i Y_i + z_i Z_i + \cdots + t_i T_i.$$
$$\therefore d_i\Delta = dx_i X_i + dy_i Y_i + dz_i Z_i + \cdots + dt_i T_i.$$

There must be n such expressions for the total differential, each of which is obviously Δ, after changing the elements of the ith row into their differentials.

$$\therefore [d\Delta]^* = \begin{vmatrix} dx_1 & dy_1 & dz_1 & \dots & dt_1 \\ x_2 & y_2 & z_2 & \dots & t_2 \\ \dots & \dots & \dots & \dots & \dots \\ x_n & y_n & z_n & \dots & t_n \end{vmatrix} + \begin{vmatrix} x_1 & y_1 & z_1 & \dots & t_1 \\ dx_2 & dy_2 & dz_2 & \dots & dt_2 \\ \dots & \dots & \dots & \dots & \dots \\ x_n & y_n & z_n & \dots & t_n \end{vmatrix}$$
$$+ \cdots + \begin{vmatrix} x_1 & y_1 & z_1 & \dots & t_1 \\ x_2 & y_2 & z_2 & \dots & t_2 \\ \dots & \dots & \dots & \dots & \dots \\ dx_n & dy_n & dz_n & \dots & dt_n \end{vmatrix}.$$

From the differentials, partial or total, we, of course, pass to the corresponding derivatives in the usual way.

Illustrations.

$$v = \frac{M}{N};\ N^2 dv = \begin{vmatrix} dM & M \\ dN & N \end{vmatrix}. \quad d \begin{vmatrix} dM & M \\ dN & N \end{vmatrix} = \begin{vmatrix} d^2M & M \\ d^2N & N \end{vmatrix}.$$

Let $$| a_{11} a_{22} a_{33} a_{44} | \equiv \Delta \equiv \begin{vmatrix} a & 2b & c & 0 \\ 0 & a & 2b & c \\ b & 2c & k & 0 \\ 0 & b & 2c & k \end{vmatrix},$$

$$\frac{d\Delta}{da} = A_{11} + A_{22} = \begin{vmatrix} a & 2b & c \\ 2c & k & 0 \\ b & 2c & k \end{vmatrix} + k \begin{vmatrix} a & c \\ b & k \end{vmatrix}$$

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* The [ ] denote the total differential.

$$\frac{d\Delta}{db}=2A_{12}+2A_{23}+A_{31}+A_{42}=4b\begin{vmatrix} b & c \\ c & k \end{vmatrix}-4k\begin{vmatrix} a & b \\ b & c \end{vmatrix}$$

$$+\begin{vmatrix} 2b & c & 0 \\ a & 2b & c \\ b & 2c & k \end{vmatrix}-c\begin{vmatrix} a & c \\ b & k \end{vmatrix}.$$

$$\frac{d\Delta}{dc}=A_{13}+A_{24}+2A_{32}+2A_{43}=\cdots,$$

$$\frac{d\Delta}{dk}=A_{33}+A_{44}=\cdots.$$

66. THEOREM. — *If the elements of Δ are all functions of the same variable x, $\frac{d\Delta}{dx}$ equals the sum of n determinants, each of which is obtained from Δ by substituting the derivatives of the elements of a row for the elements themselves.*

The truth of this proposition is evident from the preceding. Thus, if

$$\Delta\equiv\begin{vmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ f_{21}(x) & f_{22}(x) & \cdots & f_{2n}(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1}(x) & f_{n2}(x) & \cdots & f_{nn}(x) \end{vmatrix},$$

$$\frac{d\Delta}{dx}=\begin{vmatrix} f_{11}'(x) & f_{12}'(x) & \cdots & f_{1n}'(x) \\ f_{21}(x) & f_{22}(x) & \cdots & f_{2n}(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1}(x) & f_{n2}(x) & \cdots & f_{nn}(x) \end{vmatrix}+\begin{vmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ f_{21}'(x) & f_{22}'(x) & \cdots & f_{2n}'(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1}(x) & f_{n2}(x) & \cdots & f_{nn}(x) \end{vmatrix}+\cdots$$

$$+\begin{vmatrix} f_{11}(x) & f_{12}(x) & \cdots & f_{1n}(x) \\ f_{21}(x) & f_{22}(x) & \cdots & f_{2n}(x) \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1}'(x) & f_{n2}'(x) & \cdots & f_{nn}'(x) \end{vmatrix}.$$

If
$$\Delta\equiv\begin{vmatrix} 1 & \lambda_1x & 0 & 0 \\ 1 & 1 & \lambda_2x & 0 \\ 0 & 1 & 1 & \lambda_3x \\ 0 & 0 & 1 & 1 \end{vmatrix}=\lambda_1\lambda_2\lambda_3\begin{vmatrix} \frac{1}{\lambda_1} & x & 0 & 0 \\ \frac{1}{\lambda_2} & \frac{1}{\lambda_2} & x & 0 \\ 0 & \frac{1}{\lambda_3} & \frac{1}{\lambda_3} & x \\ 0 & 0 & 1 & 1 \end{vmatrix},$$

the student may show that

$$\frac{d\Delta}{dx} = -\lambda_1\lambda_2\lambda_3 \left[\begin{vmatrix} \frac{1}{\lambda_2} & x & 0 \\ 0 & \frac{1}{\lambda_3} & x \\ 0 & 1 & 1 \end{vmatrix} + \begin{vmatrix} \frac{1}{\lambda_1} & x & 0 \\ 0 & \frac{1}{\lambda_3} & x \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} \frac{1}{\lambda_1} & x & 0 \\ \frac{1}{\lambda_2} & \frac{1}{\lambda_2} & x \\ 0 & 0 & 1 \end{vmatrix} \right].$$

CHAPTER III.

APPLICATIONS AND SPECIAL FORMS.

67. We have now discussed the origin and some of the properties of determinants; it remains to show how useful these functions are in application, and to examine some of the *Special Forms* that are of frequent occurrence. Within the limits of an elementary work like this it will be possible to select only a very few of the many important applications, and to touch somewhat briefly upon the special forms. Enough will be given, however, to enable the student to pursue his further investigations with pleasure and profit. We now return to the problem with which we commenced the presentation of determinants, and proceed to the

Solution of Linear Equations, and Elimination.

68. Consider the set of three simultaneous linear equations:

$$\left.\begin{array}{l} a_1x+b_1y+c_1z=m_1 \\ a_2x+b_2y+c_2z=m_2 \\ a_3x+b_3y+c_3z=m_3 \end{array}\right\}, \quad \text{and } \Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Multiply these equations by A_1, A_2, and A_3 respectively, and add by columns, obtaining:

$$\begin{aligned}(a_1A_1+a_2A_2+a_3A_3)x+(b_1A_1+b_2A_2+b_3A_3)y& \\ +(c_1A_1+c_2A_2+c_3A_3)z& \\ =m_1A_1+m_2A_2+m_3A_3&.\end{aligned}$$

By **45** the coefficients of y and z vanish; the coefficient of x is $\Delta \equiv |a_1\, b_2\, c_3|$, and the absolute term is $|m_1\, b_2\, c_3|$.

Whence
$$x=\frac{|m_1\ b_2\ c_3|}{|a_1\ b_2\ c_3|}.$$

If we had multiplied the given equations by B_1, B_2, B_3, we should have caused the coefficients of x and z to disappear in the resulting equation, and would have found

$$y = \frac{|a_1\ m_2\ c_3|}{|a_1\ b_2\ c_3|}.$$

Using C_1, C_2, C_3 as multipliers, we should find, similarly,

$$z = \frac{|a_1\ b_2\ m_3|}{|a_1\ b_2\ c_3|}.$$

69. To generalize the solution of the preceding article is now an easy step. Given

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1r}x_r + \cdots + a_{1n}x_n = m_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2r}x_r + \cdots + a_{2n}x_n = m_2 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rr}x_r + \cdots + a_{rn}x_n = m_r \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nr}x_r + \cdots + a_{nn}x_n = m_n \end{array}\right\}, \text{ I.,}$$

and

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2r} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} & \cdots & a_{rn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} & \cdots & a_{nn} \end{vmatrix}.$$

Here Δ is, as before, the determinant formed from the n^2 coefficients in the first members of equations I., and is called the *determinant of the system.*

Multiplying equations I. in order by A_{1r}, A_{2r}, $\ldots$ A_{rr}, $\ldots$ A_{nr}, and adding by columns, we find

$$\begin{array}{l} (a_{11}A_{1r} + a_{21}A_{2r} + \cdots + a_{r1}A_{rr} + \cdots + a_{n1}A_{nr})x_1 \\ +(a_{12}A_{1r} + a_{22}A_{2r} + \cdots + a_{r2}A_{rr} + \cdots + a_{n2}A_{nr})x_2 \\ + \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ +(a_{1r}A_{1r} + a_{2r}A_{2r} + \cdots + a_{rr}A_{rr} + \cdots + a_{nr}A_{nr})x_r \\ + \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ +(a_{1n}A_{1r} + a_{2n}A_{2r} + \cdots + a_{rn}A_{rr} + \cdots + a_{nn}A_{nr})x_n \\ = m_1A_{1r} + m_2A_{2r} + \cdots + m_rA_{rr} + \cdots + m_nA_{nr}. \end{array} \quad (A)$$

In equation (A) the coefficient of all the unknowns except the coefficient of x_r vanish, and the coefficient of x_r is obviously Δ. The second member of (A) is evidently what Δ becomes when $m_1, m_2, \dots m_n$ are put for the corresponding elements of the rth column. Hence

$$x_r = \begin{vmatrix} a_{11} & a_{12} & \dots & m_1 & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & m_2 & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{r1} & a_{r2} & \dots & m_r & \dots & a_{rn} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & m_n & \dots & a_{nn} \end{vmatrix} \div \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1r} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2r} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{r1} & a_{r2} & \dots & a_{rr} & \dots & a_{rn} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nr} & \dots & a_{nn} \end{vmatrix}.$$

Translating this formula, we have:

*The value of each of n unknowns in a set of n linear simultaneous equations is the quotient of two determinants; the divisor (denominator) is the same for all the unknowns and is the determinant Δ of the nth degree formed by writing the coefficients of the unknowns in order (i.e., the determinant of the system); the numerator of the value of any unknown as x_r is obtained from Δ by substituting for the elements of its rth column the second members of the given equations in order.**

70. The following modification of the solution already given of equations I. will be interesting. Employing the same notation as in **69**, we have

$$x_r\Delta = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1r}x_r & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2r}x_r & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{r1} & a_{r2} & \dots & a_{rr}x_r & \dots & a_{rn} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nr}x_r & \dots & a_{nn} \end{vmatrix},$$

which, by **37**,

* This is the rule for the solution of simultaneous linear equations first obtained by Leibnitz, and subsequently rediscovered by Cramer. (See opening paragraph of Chapter I.)

$$= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r-1} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1r-1}x_{r-1} + a_{1r}x_r + \cdots \\ a_{21} & a_{22} & \cdots & a_{2r-1} & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2r-1}x_{r-1} + a_{2r}x_r + \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1} & a_{r2} & \cdots & a_{rr-1} & a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rr-1}x_{r-1} + a_{rr}x_r + \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nr-1} & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nr-1}x_{r-1} + a_{nr}x_r + \cdots \end{vmatrix}$$

$$\begin{vmatrix} + a_{1n}x_n & a_{1r+1} & \cdots & a_{1n} \\ + a_{2n}x_n & a_{2r+1} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ + a_{rn}x_n & a_{rr+1} & \cdots & a_{rn} \\ \cdots & \cdots & \cdots & \cdots \\ + a_{nn}x_n & a_{nr+1} & \cdots & a_{nn} \end{vmatrix}.$$

Now substitute in the last determinant the values of the elements of the rth column, and

$$x_r\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r-1} & m_1 & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2r-1} & m_2 & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1} & a_{r2} & \cdots & a_{rr-1} & m_r & \cdots & a_{rn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nr-1} & m_n & \cdots & a_{nn} \end{vmatrix},$$

$$\therefore\ x_r = \frac{|\,a_{11}a_{22}\ldots m_r \ldots a_{nn}|}{|\,a_{11}a_{22}\ldots a_{rr}\ldots a_{nn}|},$$

as before.

A simple example of the methods of **69** and **70** is the solution of the following equations:

$$\left.\begin{aligned} 5x + 3y + 3z &= 48 \\ 2x + 6y - 3z &= 18 \\ 8x - 3y + 2z &= 21 \end{aligned}\right\}. \qquad \text{Here } \Delta \equiv \begin{vmatrix} 5 & 3 & 3 \\ 2 & 6 & -3 \\ 8 & -3 & 2 \end{vmatrix} = -231,$$

$$\therefore\ x = \frac{\begin{vmatrix} 48 & 3 & 3 \\ 18 & 6 & -3 \\ 21 & -3 & 2 \end{vmatrix}}{-231} = 3;\ y = \frac{\begin{vmatrix} 5 & 48 & 3 \\ 2 & 18 & -3 \\ 8 & 21 & 2 \end{vmatrix}}{-231} = 5;\ z = \frac{\begin{vmatrix} 5 & 3 & 48 \\ 2 & 6 & 18 \\ 8 & -3 & 21 \end{vmatrix}}{-231} = 6.$$

As another example, we may solve the equations:

$$\left.\begin{aligned} y + z + u &= a \\ z + u + x &= b \\ u + x + y &= c \\ x + y + z &= d \end{aligned}\right\}. \qquad \text{Here } \Delta \equiv \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = -3.$$

The student may show that

$$x = \tfrac{1}{3}(b + c + d - 2a)\,; \quad y = \tfrac{1}{3}(c + d + a - 2b)\,;$$
$$z = \tfrac{1}{3}(d + a + b - 2c)\,; \quad u = \tfrac{1}{3}(a + b + c - 2d).$$

71. We have hitherto tacitly assumed that neither Δ nor m_i $(i = 1, 2, \ldots n)$ should vanish. If Δ vanishes and m_i does not, the value of each unknown becomes infinite. If m_i vanishes while Δ does not, the values of the unknowns are severally zero; but when m_i vanishes, the system consists of homogeneous equations, and their solution is given later. If m_i does not vanish, but Δ and the numerators of the unknowns do vanish, then we have the following theorem.

72. *If the equations of a set are not independent*, i.e., *if any one* (*or more*) *is a consequence of the others*, *the value of each unknown takes the form* $\dfrac{0}{0}$.

Since the equations are all linear, any one can be derived from the others only by the addition of two or more of them after each has been multiplied by some constant factor. But this gives rise in the determinant numerator and denominator of the value of any unknown to two or more identical rows, and hence numerator and denominator vanish.

For an example, take

$$\left.\begin{array}{l} a_1x_1 + b_1x_2 \;\; + c_1x_3 \;\; = m_1 \\ a_2x_1 + b_2x_2 \;\; + c_2x_3 \;\; = m_2 \\ a_1^2x_1 + a_1b_1x_2 + a_1c_1x_3 = a_1m_1 \end{array}\right\}; \text{ where } \Delta \equiv a_1\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0.$$

We find

$$x_1 = a_1\frac{\begin{vmatrix} m_1 & b_1 & c_1 \\ m_2 & b_2 & c_2 \\ m_1 & b_1 & c_1 \end{vmatrix}}{\Delta} = \frac{0}{0}; \quad x_2 = a_1\frac{|a_1\, m_2\, c_1|}{\Delta} = \frac{0}{0}; \quad x_3 = a_1\frac{|a_1\, b_2\, m_1|}{\Delta} = \frac{0}{0}.$$

For a second example, the student may show that the values of the unknowns in the following equations take the form $\dfrac{0}{0}$.

$$\left.\begin{array}{r} 3x + 2y - 5z = \;\;4 \\ 6x - 3y + 4z = \;22 \\ y - 2z = -2 \end{array}\right\}.$$

73. If $m_1 = m_2 = \cdots = m_{n-1} = 0$, and one m as m_n does not, we evidently get

$$x_1 = \frac{m_n A_{n1}}{\Delta};\quad x_2 = \frac{m_n A_{n2}}{\Delta};\quad \cdots;\quad x_n = \frac{m_n A_{nn}}{\Delta}.$$

Whence $$\frac{x_1}{A_{n1}} = \frac{x_2}{A_{n2}} = \cdots \frac{x_n}{A_{nn}} = \frac{m_n}{\Delta}.$$

74. If $m_1 = m_2 = \cdots = m_n = 0$, *i.e., if equations* I. *become homogeneous, then,* unless $x_1, x_2, \cdots x_n$ *are severally zero,* Δ *must vanish.*

In that case, equations I. become

$$\left.\begin{array}{l}
l_1 \equiv a_{11}x_1 + a_{12}x_2 + \cdots + a_{1r}x_r + \cdots + a_{1n}x_n = 0 \\
l_2 \equiv a_{21}x_1 + a_{22}x_2 + \cdots + a_{2r}x_r + \cdots + a_{2n}x_n = 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
l_r \equiv a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rr}x_r + \cdots + a_{rn}x_n = 0 \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
l_n \equiv a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nr}x_r + \cdots + a_{nn}x_n = 0
\end{array}\right\} \text{II.}$$

Since $x_r\Delta = |a_{11}\, a_{22}\, a_{33} \cdots m_r \cdots a_{nn}| = 0$ (m_r being zero), the truth of the assertion is obvious.

An example is furnished by the homogeneous equations:

$$\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0
\end{array}\right\}. \tag{E}$$

Multiplying equations (E) by A_{11}, A_{21}, A_{31}, respectively, and adding by columns, we have

$$\begin{aligned}
&(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31})x_1 \\
&+ (a_{12}A_{11} + a_{22}A_{21} + a_{32}A_{31})x_2 \\
&+ (a_{13}A_{11} + a_{23}A_{21} + a_{33}A_{31})x_3 = 0.
\end{aligned}$$

The coefficients of x_2 and x_3 are zero, and we have

$$x_1\Delta = 0, \qquad \therefore \Delta = 0.$$

As a further illustration, the student may show that *if*

$$ux_1x + vy_1y + wz_1z + u_1(yz_1 + y_1z) + v_1(zx_1 + z_1x) + w_1(xy_1 + x_1y)$$

is zero for all values of x, y, *and* z, then

$$uvw - uu_1^2 - vv_1^2 - ww_1^2 + 2\,u_1v_1w_1 = 0.$$

Observe that by the given conditions the coefficients of x, y, z must severally vanish.

75. With the help of **74** we obtain an interesting proof of the multiplication theorem of **53**. Consider the simultaneous equations

$$\left.\begin{array}{llll}(a_1-\lambda)x_1 + & b_1x_2 + & c_1x_3 & = 0 \\ a_2x_1 & +(b_2-\lambda)x_2 + & c_2x_3 & = 0 \\ a_3x_1 & + \quad b_3x_2 & +(c_3-\lambda)x_3 & = 0\end{array}\right\} \text{I.}$$

By the preceding article we must have

$$\Delta \equiv \begin{vmatrix} a_1-\lambda & b_1 & c_1 \\ a_2 & b_2-\lambda & c_2 \\ a_3 & b_3 & c_3-\lambda \end{vmatrix} = 0,$$

or
$$\lambda^3 - M\lambda^2 + N\lambda - P = 0; \tag{a}$$

where we notice especially that

$$P \equiv | a_1 \; b_2 \; c_3 |.$$

Let the roots of (a) be λ_1, λ_2, λ_3; then, evidently,

$$P = -\lambda_1 \lambda_2 \lambda_3.$$

Now, from I. we obtain three new equations as follows: Multiply equations I. by α_1, α_2, α_3 respectively, and add them together; also multiply equations I. by β_1, β_2, β_3 respectively, and add; finally, multiply equations I. by γ_1, γ_2, γ_3 respectively, and add. We now have three new equations where the determinant of the system is

$$\Delta' \equiv \begin{vmatrix} a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 - \alpha_1\lambda & b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 - \alpha_2\lambda & c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 - \alpha_3\lambda \\ a_1\beta_1 + a_2\beta_2 + a_3\beta_3 - \beta_1\lambda & b_1\beta_1 + b_2\beta_2 + b_3\beta_3 - \beta_2\lambda & c_1\beta_1 + c_2\beta_2 + c_3\beta_3 - \beta_3\lambda \\ a_1\gamma_1 + a_2\gamma_2 + a_3\gamma_3 - \gamma_1\lambda & b_1\gamma_1 + b_2\gamma_2 + b_3\gamma_3 - \gamma_2\lambda & c_1\gamma_1 + c_2\gamma_2 + c_3\gamma_3 - \gamma_3\lambda \end{vmatrix} = 0,$$

or
$$Q\lambda^3 - M_1\lambda^2 + N_1\lambda - P_1 = 0, \tag{b}$$

where we observe that P_1 is what Δ' becomes when we put $\lambda=0$, and that

$$Q \equiv |\alpha_1 \ \beta_2 \ \gamma_3|.$$

Further, since

$$\frac{P_1}{Q} = -\lambda_1 \ \lambda_2 \ \lambda_3 = P,$$

it follows that

$$P_1 = PQ \equiv |a_1 \ b_2 \ c_3| \times |\alpha_1 \ \beta_2 \ \gamma_3|.$$

But P_1 is exactly the determinant obtained by **53**, and this was to be shown.

76. The condition $\Delta=0$ being fulfilled, the equations no longer determine the actual values of the unknowns; they determine only the ratios of these values. For, if x_1', x_2', ... x_n' satisfy equations II., so will kx_1', kx_2', ... kx_n', k being any factor. Any $n-1$ of the given equations will suffice in general to determine the ratios of $n-1$ of the unknowns to the remaining one. An example will make this clear. We employ for brevity only three equations:

$$\left.\begin{aligned} a_1x+b_1y+c_1z&=0 \\ a_2x+b_2y+c_2z&=0 \\ a_3x+b_3y+c_3z&=0 \end{aligned}\right\} (a).$$

Write these equations

$$\left.\begin{aligned} a_1\frac{x}{y}+c_1\frac{z}{y}&=-b_1 \\ a_2\frac{x}{y}+c_2\frac{z}{y}&=-b_2 \\ a_3\frac{x}{y}+c_3\frac{z}{y}&=-b_3 \end{aligned}\right\} (b).$$

From any two of equations (b) we may find the values of $\frac{x}{y}$, $\frac{z}{y}$; thus from the first two

$$\frac{x}{y} = -\frac{|b_1 \ c_2|}{|a_1 \ c_2|}; \quad \frac{z}{y} = -\frac{|a_1 \ b_2|}{|a_1 \ c_2|}.$$

Again, equations (b) are to be simultaneous; hence these values of $\frac{x}{y}$ and $\frac{z}{y}$ must satisfy the third equation.

Substituting,

$$a_3|b_1\ c_2| - b_3|a_1\ c_2| + c_3|a_1\ b_2| = 0;$$

or

$$\Delta = 0.$$

Since from the preceding equations $\frac{x}{y}$ also equals

$$-\frac{|b_1\ c_3|}{|a_1\ c_3|} = -\frac{|b_2\ c_3|}{|a_2\ c_3|},$$

we have

$$\frac{x}{y} = \frac{A_1}{B_1} = \frac{A_2}{B_2} = \frac{A_3}{B_3}.$$

In the same way,

$$\frac{z}{y} = \frac{C_1}{B_1} = \frac{C_2}{B_2} = \frac{C_3}{B_3},$$

and hence

$$\frac{x}{z} = \frac{A_1}{C_1} = \frac{A_2}{C_2} = \frac{A_3}{C_3};$$

or,

$$\begin{aligned} x : y : z &= A_1 : B_1 : C_1 \\ &= A_2 : B_2 : C_2 \\ &= A_3 : B_3 : C_3. \end{aligned}$$

That is to say, *The ratio of any two unknowns in a set of homogeneous equations is equal to the ratio of the cofactors in* Δ *of the coefficients of these unknowns in any of the given equations.*

The general proof of the proposition just stated may be given as follows. We have to show (equations II.) that

$$\begin{aligned} x_1 : x_2 : x_3 : \cdots : x_r : \cdots : x_n &= A_{11} : A_{12} : A_{13} : \cdots : A_{1r} : \cdots : A_{1n} \\ &= A_{21} : A_{22} : A_{23} : \cdots : A_{2r} : \cdots : A_{2n} \\ &= \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots\quad \cdots \\ &= A_{n1} : A_{n2} : A_{n3} : \cdots : A_{nr} : \cdots : A_{nn}. \end{aligned}$$

If these proportions are true, we must have the equations

$$x_r = \lambda A_{pr} \qquad (\lambda = \text{constant};\ p = 1, 2, \cdots n). \qquad (E_1)$$

The equation

$$a_{r1}A_{p1} + a_{r2}A_{p2} + \cdots + a_{rr}A_{pr} + \cdots + a_{rn}A_{pn} = 0 \qquad (E_2)$$

is always true, whatever the value of p, since Δ is itself zero.

Substituting in (E_2) the values of

$$A_{p1},\ A_{p2}, \cdots A_{pn},$$

as obtained from (E_1), and multiplying by λ, there results

$$a_{r1}x_1 + a_{r2}x_2 + a_{r3}x_3 + \cdots + a_{rr}x_r + \cdots + a_{rn}x_n = 0.$$

This last being a true equation, the proportions from which it is derived must hold.*

77. From the last article, or the two preceding articles, we deduce the important conclusion. *In order that n linear homogeneous equations may be simultaneous, it is necessary and sufficient that the determinant of the system vanishes.* In that case any one of the equations is expressible linearly in terms of all the others, provided the first minors A_{ik} do not all vanish. For we have in general, Δ being zero, and $l_1, l_2, \ldots l_n$ representing the linear functions of equations II.,

$$l_1A_{1k} + l_2A_{2k} + \cdots + l_nA_{nk} = 0;$$

hence, if one at least of the first minors $A_{1k}, A_{2k}, \ldots A_{nk}$ is not zero, as for example A_{1k}, l_1 must be expressible linearly in terms of $l_2, l_3, \ldots l_n$, and hence $l_1 = 0$ is superfluous. If all the first minors vanish, and one at least of the second minors does not, then, similarly, it may be shown that *two* equations are superfluous, the system being doubly indeterminate, and so on.

78. Among the proportions of article **76** consider the following:

$$x_1 : x_2 : x_3 : \cdots x_n = A_{n1} : A_{n2} : A_{n3} : \cdots A_{nn}. \qquad (P)$$

---

* This demonstration applies of course so long as the first minors of Δ do not all vanish.

$A_{n1}, A_{n2}, A_{n3}, \cdots A_{nn}$ are, none of them, functions of the coefficients of the last equation of set II. in **74**,

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n = 0.$$

Hence, proportions (P) give the ratios of the unknowns $x_1, x_2, x_3, \cdots x_n$, that satisfy the $n-1$ equations

$$\left.\begin{array}{llllll} a_{11}x_1 & + a_{12}x_2 & + \cdots & + a_{1n}x_n & = 0 \\ a_{21}x_1 & + a_{22}x_2 & + \cdots & + a_{2n}x_n & = 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-11}x_1 & + a_{n-12}x_2 & + \cdots & + a_{n-1n}x_n & = 0 \end{array}\right\} \text{III.},$$

if we denote by A_{n1} the determinant formed from the coefficients in equations III. after suppressing the first column of terms, by A_{n2} the determinant formed from the coefficients of equations III. after suppressing the second column of terms, and so on. Hence having given n homogeneous equations containing $n+1$ unknowns

$$\left.\begin{array}{llllll} a_{11}x_1 & + a_{12}x_2 & + \cdots & + a_{1n+1}x_{n+1} & = 0 \\ a_{21}x_1 & + a_{22}x_2 & + \cdots & + a_{2n+1}x_{n+1} & = 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}x_1 & + a_{n2}x_2 & + \cdots & + a_{nn+1}x_{n+1} & = 0 \end{array}\right\} \text{IV.},$$

we find the ratios of the unknowns as follows:

put
$$\Delta_i \equiv (-1)^{i-1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1i-1} & a_{1i+1} & \cdots & a_{1n+1} \\ a_{21} & a_{22} & \cdots & a_{2i-1} & a_{2i+1} & \cdots & a_{2n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{ni-1} & a_{ni+1} & \cdots & a_{nn+1} \end{vmatrix}.$$

Then from what precedes

$$x_1 : x_2 : x_3 : \cdots : x_{n+1} = \Delta_1 : \Delta_2 : \Delta_3 : \cdots : \Delta_{n+1}.$$

79. Consider the following n equations containing $n-1$ unknowns.

$$\left.\begin{array}{lllllll} a_{11}x_1 & + a_{12}x_2 & + \cdots & + a_{1n-1}x_{n-1} & + p_1 & = 0 \\ a_{21}x_1 & + a_{22}x_2 & + \cdots & + a_{2n-1}x_{n-1} & + p_2 & = 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-11}x_1 & + a_{n-12}x_2 & + \cdots & + a_{n-1n-1}x_{n-1} & + p_{n-1} & = 0 \\ a_{n1}x_1 & + a_{n2}x_2 & + \cdots & + a_{nn-1}x_{n-1} & + p_n & = 0 \end{array}\right\} \text{V.}$$

Equations V. may be made homogeneous by multiplying them by u, and regarding $x_1u, x_2u, \ldots x_nu, u$, as the unknowns, u being any arbitrary quantity. Whence, if these equations are simultaneous, we have by **77**

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & p_1 \\ a_{21} & a_{22} & \cdots & a_{2n-1} & p_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-1} & p_{n-1} \\ a_{n1} & a_{n2} & \cdots & a_{nn-1} & p_n \end{vmatrix} = 0.$$

This result may be expressed as follows: *n equations (not homogeneous) containing $n-1$ unknowns are simultaneous if the determinant of the nth degree formed from all the coefficients (the second members of the equations being included among these coefficients) vanishes.*

This condition could also be derived from equations II., Art. **74**, by putting $x_n = 1$. Those equations, n in number, then contain $n-1$ unknowns; and if the equations are simultaneous, we see that $|a_{1n}|$ must vanish.

80. With the help of the preceding article another solution of a set of linear equations may be obtained. For brevity we employ only three equations:

$$\left.\begin{aligned} &(1)\quad a_1x_1 + b_1x_2 + c_1x_3 = m_1 \\ &(2)\quad a_2x_1 + b_2x_2 + c_2x_3 = m_2 \\ &(3)\quad a_3x_1 + b_3x_2 + c_3x_3 = m_3 \end{aligned}\right\}.$$

Take with these equations another,

$$(4)\quad a_4x_1 + b_4x_2 + c_4x_3 = m_4,$$

which we suppose consistent with the first three, and in which a_4, b_4, c_4, m_4 are undetermined. By **79**

$$|a_1\ b_2\ c_3\ m_4| = 0;$$

or,

$$a_4A_4 + b_4B_4 + c_4C_4 + m_4M_4 = 0; \qquad (5)$$

where, as usual,

$$A_4 = -|b_1\ c_2\ m_3|; \quad B_4 = |a_1\ c_2\ m_3|;$$
$$C_4 = -|a_1\ b_2\ m_3|; \quad M_4 = |a_1\ b_2\ c_3| \equiv \Delta.$$

Now if we eliminate m_4 from equations (4) and (5), we get

$$a_4\left(x_1+\frac{A_4}{\Delta}\right)+b_4\left(x_2+\frac{B_4}{\Delta}\right)+c_4\left(x_3+\frac{C_4}{\Delta}\right)=0. \qquad (6)$$

Since equation (6) must be true whatever the values of a_4, b_4, c_4, the coefficients of a_4, b_4, c_4 severally vanish.

$$\therefore\ x_1=-\frac{A_4}{\Delta}; \qquad x_2=-\frac{B_4}{\Delta}; \qquad x_3=-\frac{C_4}{\Delta};$$

or,
$$x_1=\frac{|m_1\ b_2\ c_3|}{\Delta}; \quad x_2=\frac{|a_1\ m_2\ c_3|}{\Delta}; \quad x_3=\frac{|a_1\ b_2\ m_3|}{\Delta}.$$

81. Let us now return to equations I. Art. **69**. Considering $m_1, m_2, m_3, \ldots m_n$ as linear functions of the x's, we can express any new linear function

$$c_1x_1+c_2x_2+\cdots+c_nx_n=y$$

in terms of the m's.

Thus, if we have given

$$\left.\begin{array}{lllll} c_1x_1 & +c_2x_2 & +\ \cdots & +c_nx_n & =y \\ a_{11}x_1 & +a_{12}x_2 & +\ \cdots & +a_{1n}x_n & =m_1 \\ a_{21}x_1 & +a_{22}x_2 & +\ \cdots & +a_{2n}x_n & =m_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}x_1 & +a_{n2}x_2 & +\ \cdots & +a_{nn}x_n & =m_n \end{array}\right\}\ \text{VI.},$$

by **79**,

$$\Delta'\equiv\begin{vmatrix} c_1 & c_2 & \cdots & c_n & y \\ a_{11} & a_{12} & \cdots & a_{1n} & m_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & m_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & m_n \end{vmatrix}=0.$$

Now if $\Delta\equiv|a_{1n}|$, we readily obtain

$$\Delta'\pm y\Delta=\pm y\Delta;$$

or,
$$\pm y\Delta=\begin{vmatrix} c_1 & c_2 & \cdots & c_n & 0 \\ a_{11} & a_{12} & \cdots & a_{1n} & m_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & m_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & m_n \end{vmatrix}.$$

82. We have seen that if n homogeneous equations are to be consistent with each other (simultaneous), the determinant of the system must vanish. The equation

$$\Delta = 0$$

then is an equation of relation between the coefficients, and is really the result of eliminating the unknowns from the given equations. We shall soon investigate this resulting equation of condition or resultant in detail. We here deduce a general form by which the result of eliminating n unknowns from p given linear equations, supposed simultaneous, may be expressed, p being greater than n.

Given

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{p1}x_1 + a_{p2}x_2 + \ldots + a_{pn}x_n = 0 \end{array}\right\} \text{VII.}$$

If these equations are to be satisfied for other than zero values of the variables, the determinant of the system for any n of them must vanish by **77**. The equation expressing this condition is obtained by writing

$$\left\Vert \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1} & a_{p2} & \cdots & a_{pn} \end{array}\right\Vert = 0. \qquad (M)$$

Equation (M) is accordingly interpreted to mean that every determinant of the nth order formed from any n rows of the matrix on the left must vanish. For an example the student may eliminate the two ratios $x_1 : x_2 : x_3$ from the five equations

$$a_i x_1 + b_i x_2 + c_i x_3 = 0 \quad (i = 1, 2, \ldots 5),$$

obtaining the equation

$$\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \\ a_5 & b_5 & c_5 \end{Vmatrix} = 0, \text{ or } \begin{Vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \end{Vmatrix} = 0.$$

83. Suppose we have given

$$\begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 + d_1x_4 &= 0, \\ a_2x_1 + b_2x_2 + c_2x_3 + d_2x_4 &= 0, \\ a_3x_1 + b_3x_2 + c_3x_3 + d_3x_4 &= 0\,; \end{aligned}$$

then, by **78**,

$$x_1 : x_2 : x_3 : x_4 = |b_1\ c_2\ d_3| : -|a_1\ c_2\ d_3| : |a_1\ b_2\ d_3| : -|a_1\ b_2\ c_3|.$$

Substituting the values of

$$\frac{x_1}{x_3},\quad \frac{x_2}{x_3},\quad \frac{x_4}{x_3},$$

we get the relations

$$\begin{aligned} a_1 |b_1\ c_2\ d_3| - b_1 |a_1\ c_2\ d_3| + c_1 |a_1\ b_2\ d_3| - d_1 |a_1\ b_2\ c_3| &= 0, \\ a_2 |b_1\ c_2\ d_3| - b_2 |a_1\ c_2\ d_3| + c_2 |a_1\ b_2\ d_3| - d_2 |a_1\ b_2\ c_3| &= 0, \\ a_3 |b_1\ c_2\ d_3| - b_3 |a_1\ c_2\ d_3| + c_3 |a_1\ b_2\ d_3| - d_3 |a_1\ b_2\ c_3| &= 0, \end{aligned}$$

which are all expressed by the matrix

$$\begin{Vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{Vmatrix}.$$

To generalize this, we return to art. **78**.

From equations IV. we found

$$x_1 : x_2 : x_3 : \cdots x_{n+1} = \Delta_1 : \Delta_2 : \Delta_3 \cdots \Delta_{n+1}.$$

Substituting in equations IV. the values of

$$\frac{x_1}{x_r},\quad \frac{x_2}{x_r}, \cdots \frac{x_{n+1}}{x_r},$$

given by these proportions, we have

$$\left.\begin{array}{l} a_{11}\Delta_1 + a_{12}\Delta_2 + \cdots + a_{1r}\Delta_r + \cdots + a_{1n+1}\Delta_{n+1} = 0 \\ a_{21}\Delta_1 + a_{22}\Delta_2 + \cdots + a_{2r}\Delta_r + \cdots + a_{2n+1}\Delta_{n+1} = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{r1}\Delta_1 + a_{r2}\Delta_2 + \cdots + a_{rr}\Delta_r + \cdots + a_{rn+1}\Delta_{n+1} = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{n1}\Delta_1 + a_{n2}\Delta_2 + \cdots + a_{nr}\Delta_r + \cdots + a_{nn+1}\Delta_{n+1} = 0 \end{array}\right\}. \quad (R)$$

These n relations are expressed by the matrix

$$\left\| \begin{array}{ccccccc} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1n} & a_{1n+1} \\ a_{21} & a_{22} & \cdots & a_{2r} & \cdots & a_{2n} & a_{2n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} & \cdots & a_{rn} & a_{rn+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} & \cdots & a_{nn} & a_{nn+1} \end{array} \right\| \equiv M.$$

We have accordingly, in general, from a matrix of the form M, the following relations:

$$a_{r1}\begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1r} & \cdots & a_{1n} & a_{1n+1} \\ a_{22} & a_{23} & \cdots & a_{2r} & \cdots & a_{2n} & a_{2n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r2} & a_{r3} & \cdots & a_{rr} & \cdots & a_{rn} & a_{rn+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n2} & a_{n3} & \cdots & a_{nr} & \cdots & a_{nn} & a_{nn+1} \end{vmatrix} - a_{r2}\begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1r} & \cdots & a_{1n} & a_{1n+1} \\ a_{21} & a_{23} & \cdots & a_{2r} & \cdots & a_{2n} & a_{2n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1} & a_{r3} & \cdots & a_{rr} & \cdots & a_{rn} & a_{rn+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n3} & \cdots & a_{nr} & \cdots & a_{nn} & a_{nn+1} \end{vmatrix}$$

$$+ - \cdots + (-1)^n a_{rn+1}\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1r} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2r} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} & \cdots & a_{rn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nr} & \cdots & a_{nn} \end{vmatrix} = 0,$$

in which r has successively all values from 1 to n inclusive.

84. We will now select a few examples to illustrate the foregoing processes from the vast field of application.

I. To find the condition that three right lines shall pass through the same point.

concurrent

Let
$$\left.\begin{aligned} a_1x+b_1y+c_1&=0\\ a_2x+b_2y+c_2&=0\\ a_3x+b_3y+c_3&=0\end{aligned}\right\} \qquad (A)$$

be the equations of the lines in cartesian co-ordinates, and let x_1, y_1 be the given point. Equations (A) must be satisfied for $x=x_1$, $y=y_1$; hence

$$\left.\begin{aligned} a_1x_1+b_1y_1+c_1&=0\\ a_2x_1+b_2y_1+c_2&=0\\ a_3x_1+b_3y_1+c_3&=0\end{aligned}\right\}. \qquad (B)$$

But in that case, by **79**,

$$|a_1b_2c_3|=0,$$

which expresses the required condition.

II. To find the condition that three points shall lie on the same right line.

Let
$$(x_1,\ y_1),\quad (x_2,\ y_2),\quad (x_3,\ y_3)$$

be the given points, and

$$a_1x+b_1y+c_1=0$$

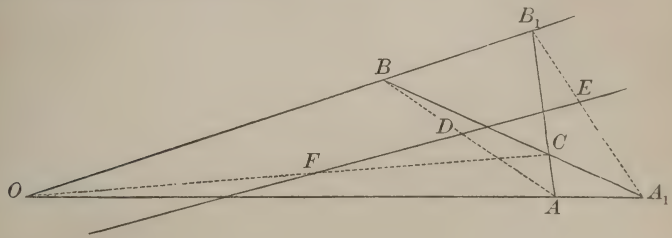
the equation of the line. Then

$$\begin{aligned} a_1x_1+b_1y_1+c_1&=0,\\ a_2x_2+b_2y_2+c_2&=0,\\ a_3x_3+b_3y_3+c_3&=0.\end{aligned}$$

Whence the required condition is

$$\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=0. \qquad (R)$$

As an application of the present example, we show that *the middle points of the three diagonals of a complete quadrilateral lie on the same straight line.*



The three diagonals being OC, BA, B_1A_1, and their middle points F, D, E, we have to show that F, D, E are on the same right line.

Take the vertex O as origin, and the sides OA_1, OB_1 as axes of reference.

Put $$a_1 = OA,\ \ a_2 = OA_1,\ \ b_1 = OB,\ \ b_2 = OB_1.$$

The co-ordinates of D are $\frac{a_1}{2}, \frac{b_1}{2}$, and the co-ordinates of E are $\frac{a_2}{2}, \frac{b_2}{2}$. The abscissa of F is half the abscissa of C, and the ordinate of F is half the ordinate of C. Hence we have to find the co-ordinates of C. The equations of AB_1 and A_1B are respectively

$$\frac{x}{a_1} + \frac{y}{b_2} = 1, \quad \text{or} \quad b_2x + a_1y = a_1b_2;$$

$$\frac{x}{a_2} + \frac{y}{b_1} = 1, \quad \text{or} \quad b_1x + a_2y = a_2b_1.$$

Whence the co-ordinates of C are

$$x = \frac{\begin{vmatrix} a_1b_2 & a_1 \\ a_2b_1 & a_2 \end{vmatrix}}{\begin{vmatrix} b_2 & a_1 \\ b_1 & a_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} b_2 & a_1b_2 \\ b_1 & a_2b_1 \end{vmatrix}}{\begin{vmatrix} b_2 & a_1 \\ b_1 & a_2 \end{vmatrix}};$$

and the co-ordinates of F are

$$\frac{a_1 a_2 (b_2 - b_1)}{2(b_2 a_2 - b_1 a_1)}, \quad \frac{b_1 b_2 (a_2 - a_1)}{2(b_2 a_2 - b_1 a_1)}.$$

Now, by equation (R) above,

$$\Delta \equiv \begin{vmatrix} \frac{a_1}{2} & \frac{b_1}{2} & 1 \\ \frac{a_2}{2} & \frac{b_2}{2} & 1 \\ \frac{a_1 a_2 (b_2 - b_1)}{2(b_2 a_2 - b_1 a_1)} & \frac{b_1 b_2 (a_2 - a_1)}{2(b_2 a_2 - b_1 a_1)} & 1 \end{vmatrix} = 0,$$

if the three points are on the same straight line.

$$\Delta = \frac{1}{4(b_2 a_2 - b_1 a_1)} \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_1 a_2 (b_2 - b_1) & b_1 b_2 (a_2 - a_1) & b_2 a_2 - b_1 a_1 \end{vmatrix}.$$

Now add the third column of this determinant multiplied by $-a_1$ to the first column; also add the third column multiplied by $-b_1$ to the second column. Then

$$\Delta = \frac{1}{4(b_2 a_2 - b_1 a_1)} \begin{vmatrix} 0 & 0 & 1 \\ a_2 - a_1 & b_2 - b_1 & 1 \\ a_1 b_1 (a_1 - a_2) & a_1 b_1 (b_1 - b_2) & b_2 a_2 - b_1 a_1 \end{vmatrix},$$

which is obviously zero. Hence F, D, E are on the same right line.

III. *To obtain the equation of a circle passing through three given points.*

The general equation of the circle is

$$(x^2 + y^2) + 2ax + 2by + c = 0.$$

If (x_1, y_1), (x_2, y_2), (x_3, y_3) are the given points,

$$(x_1^2 + y_1^2) + 2ax_1 + 2by_1 + c = 0,$$
$$(x_2^2 + y_2^2) + 2ax_2 + 2by_2 + c = 0,$$
$$(x_3^2 + y_3^2) + 2ax_3 + 2by_3 + c = 0.$$

These four equations are simultaneous for the parameters a, b, c; hence, by **79**,

$$\begin{vmatrix} x^2+y^2 & 2x & 2y & 1 \\ x_1^2+y_1^2 & 2x_1 & 2y_1 & 1 \\ x_2^2+y_2^2 & 2x_2 & 2y_2 & 1 \\ x_3^2+y_3^2 & 2x_3 & 2y_3 & 1 \end{vmatrix} = 0, \qquad (C)$$

which is the equation sought.

That equation C is the required equation of the circle determined by (x_1, y_1), (x_2, y_2), (x_3, y_3), is obvious from the form of the first member. The determinant when expanded obviously gives a function of the second degree, and having the characteristics which distinguish the equation of the circle. Moreover, this equation is satisfied for $x = x_1$, $y = y_1$, since in that case the determinant vanishes. The same is true if $x = x_2$, $y = y_2$, or $x = x_3$, $y = y_3$.

IV. *To find the relation connecting the mutual distances of four points on the circle.*

We must have, if the points are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), a determinant equation just like the last one above, except that the first row of the determinant will have the subscripts $_1$, the second row the subscripts $_2$, and so on, the last row having the subscripts $_4$.

Accordingly, multiplying together

$$\begin{vmatrix} x_1^2+y_1^2 & -2x_1 & -2y_1 & 1 \\ x_2^2+y_2^2 & -2x_2 & -2y_2 & 1 \\ x_3^2+y_3^2 & -2x_3 & -2y_3 & 1 \\ x_4^2+y_4^2 & -2x_4 & -2y_4 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2 \\ 1 & x_4 & y_4 & x_4^2+y_4^2 \end{vmatrix},$$

which are two different forms of the first member of equation (C) above, we obtain the required relation

$$\begin{vmatrix} 0 & (12)^2 & (13)^2 & (14)^2 \\ (12)^2 & 0 & (23)^2 & (24)^2 \\ (13)^2 & (23)^2 & 0 & (34)^2 \\ (14)^2 & (24)^2 & (34)^2 & 0 \end{vmatrix} = 0,$$

in which

$$(12)^2 \equiv (x_1 - x_2)^2 + (y_1 - y_2)^2, \quad (13)^2 \equiv (x_1 - x_3)^2 + (y_1 - y_3)^2,$$

and, in general, $(ik)^2$ is the square of the distance between the ith and kth points.

Expanding this determinant by **63**, III., and adding and subtracting $4(12)^2(13)^2(24)^2(34)^2$, we obtain

$$[(12)^2(34)^2 + (13)^2(24)^2 - (14)^2(23)^2]^2$$
$$- 4(12)^2(13)^2(24)^2(34)^2 = 0.$$

Whence

$$\{[(12)(34) - (13)(24) - (14)(23)]$$
$$[(12)(34) - (13)(24) + (14)(23)]\}$$
$$\times \{[(12)(34) + (13)(24) - (14)(23)]$$
$$[(12)(34) + (13)(24) + (14)(23)]\} = 0,$$

or
$$(12)(34) \pm (13)(24) \pm (14)(23) = 0,$$

which expresses the condition sought in its simplest form.

V. *To find the condition that two given straight lines in space may intersect.*

(*a*) Let
$$\frac{x-\alpha}{a_1} = \frac{y-\beta}{b_1} = \frac{z-\gamma}{c_1}, \tag{1}$$

$$\frac{x-\alpha_1}{a_2} = \frac{y-\beta_1}{b_2} = \frac{z-\gamma_1}{c_2} \tag{2}$$

be the equations of the lines. If these lines intersect, the plane

$$px + qy + rz = d$$

may be passed through them, and we must have for the first line

$$\left.\begin{aligned} p\alpha + q\beta + r\gamma &= d \quad (3) \\ pa_1 + qb_1 + rc_1 &= 0 \quad (4) \end{aligned}\right\},$$

and for the second line

$$\left.\begin{aligned} p\alpha_1 + q\beta_1 + r\gamma_1 &= d \quad (5) \\ pa_2 + qb_2 + rc_2 &= 0 \quad (6) \end{aligned}\right\}.$$

From (3) and (5)

$$p(\alpha-\alpha_1)+q(\beta-\beta_1)+r(\gamma-\gamma_1)=0. \qquad (7)$$

(4), (6), (7) being simultaneous, the required condition is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ \alpha-\alpha_1 & \beta-\beta_1 & \gamma-\gamma_1 \end{vmatrix}=0.$$

(*b*) *If the straight lines are given by the equations*

$$\left.\begin{matrix} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \end{matrix}\right\}, \qquad (1)$$

$$\left.\begin{matrix} a_3x+b_3y+c_3z=d_3 \\ a_4x+b_4y+c_4z=d_4 \end{matrix}\right\}, \qquad (2)$$

these four equations are simultaneous for the point of intersection (x, y, z), and the condition of intersection is

$$|a_1b_2c_3d_4|=0.$$

VI. *To find the equation of a plane passing through three given points* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

Let the plane be

$$a_1x+b_1y+c_1z=d_1. \qquad (1)$$

We must have

$$\left.\begin{matrix} a_1x_1+b_1y_1+c_1z_1=d_1 \\ a_1x_2+b_1y_2+c_1z_2=d_1 \\ a_1x_3+b_1y_3+c_1z_3=d_1 \end{matrix}\right\}. \qquad (D)$$

Equations (*D*) and (1) being simultaneous for the parameters a_1, b_1, c_1, d_1, we have for the equation sought

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=0.$$

VII. An interesting application of determinants is afforded by the following problems.

(*a*) *To extend a recurring series of the rth order without knowing the scale of relation.*

As is well known, a series of the form

$$u_0 + u_1 x + u_2 x^2 + \cdots + u_{n-r} x^{n-r} + \cdots + u_n x^n + \cdots$$

is a recurring series if the relation of any $r+1$ consecutive coefficients $u_n, u_{n-1}, \cdots u_{n-r}$ can be expressed by a linear equation (the scale of relation). Under these conditions the series is called a recurring series of the rth order. Every such series is accordingly determined when $2r$ of its consecutive terms are known. If all the coefficients, with the exception of the $2r$th, are known, this last is easily found. By the conditions of a recurring series

$$\left.\begin{array}{l} u_r + p_1 u_{r-1} + p_2 u_{r-2} + p_3 u_{r-3} + \cdots + p_{r-1} u_1 + p_r u_0 = 0 \\ u_{r+1} + p_1 u_r + p_2 u_{r-1} + p_3 u_{r-2} + \cdots + p_{r-1} u_2 + p_r u_1 = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ u_{2r-1} + p_1 u_{2r-2} + p_2 u_{2r-3} + p_3 u_{2r-4} + \cdots + p_{r-1} u_r + p_r u_{r-1} = 0 \\ u_{2r} + p_1 u_{2r-1} + p_2 u_{2r-2} + p_3 u_{2r-3} + \cdots + p_{r-1} u_{r+1} + p_r u_r = 0 \end{array}\right\}. \quad (F)$$

Now, by **79**,

$$\begin{vmatrix} u_r & u_{r-1} & u_{r-2} & u_{r-3} & \cdots & u_1 & u_0 \\ u_{r+1} & u_r & u_{r-1} & u_{r-2} & \cdots & u_2 & u_1 \\ u_{r+2} & u_{r+1} & u_r & u_{r-1} & \cdots & u_3 & u_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{2r-1} & u_{2r-2} & u_{2r-3} & u_{2r-4} & \cdots & u_r & u_{r-1} \\ u_{2r} & u_{2r-1} & u_{2r-2} & u_{2r-3} & \cdots & u_{r+1} & u_r \end{vmatrix} = 0,$$

whence u_{2r} is found by expanding the determinant and solving the equation.

To find u_{2r+1} we have only to increase each subscript by unity.

Applying the above process to extend the series

$$1 + x + 5x^2 + 13x^3 + \cdots,$$

we find

$$\begin{vmatrix} 5 & 1 & 1 \\ 13 & 5 & 1 \\ u_4 & 13 & 5 \end{vmatrix} = 0; \quad \begin{vmatrix} 13 & 5 & 1 \\ u_4 & 13 & 5 \\ u_5 & u_4 & 13 \end{vmatrix} = 0; \quad \begin{vmatrix} u_4 & 13 & 5 \\ u_5 & u_4 & 13 \\ u_6 & u_5 & u_4 \end{vmatrix} = 0;$$

whence $u_4 = 41$, $u_5 = 121$, $u_6 = 365$. The series is accordingly

$$1 + x + 5x^2 + 13x^3 + 41x^4 + 121x^5 + 365x^6 + \cdots.$$

(*b*) *To find the generating function for any given recurring series.*

Since a recurring series is always the quotient of two integral functions, of which the divisor is of a degree higher by 1 than the dividend, we may find the required generating function by indeterminate coefficients, as follows:

Assume the given series

$$u_0 + u_1x + u_2x^2 + \cdots + u_rx^r = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_{r-1}x^{r-1}}{1 + p_1x + p_2x^2 + \cdots + p_rx^r} \qquad (T)$$

(after both terms of the fraction have been divided by the first term of the denominator).

From the first r of equations (F) of the preceding example we can determine the constants $p_1, p_2 \cdots p_r$. We may therefore find the scale of relation. We have from equations (F), after obvious interchanges of columns,

$$p_r = \frac{\begin{vmatrix} -u_r & u_1 & \cdots & u_{r-2} & u_{r-1} \\ -u_{r+1} & u_2 & \cdots & u_{r-1} & u_r \\ -u_{r+2} & u_3 & \cdots & u_r & u_{r+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ -u_{2r-1} & u_r & \cdots & u_{2r-3} & u_{2r-2} \end{vmatrix}}{\begin{vmatrix} u_0 & u_1 & \cdots & u_{r-2} & u_{r-1} \\ u_1 & u_2 & \cdots & u_{r-1} & u_r \\ u_2 & u_3 & \cdots & u_r & u_{r+1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ u_{r-1} & u_r & \cdots & u_{2r-3} & u_{2r-2} \end{vmatrix}}.$$

Having determined the constants $p_1, p_2 \cdots p_r$, we need only clear equation (T) of fractions; and then, equating the co-

efficients of like powers of x, obtain the usual linear equations from which $a_0, a_1, a_2 \ldots a_{r-1}$ are found.

For an example, let us find the generating function of the series we extended in the last example.

Put

$$1 + x + 5x^2 + 13x^3 + 41x^4 + \cdots \equiv \frac{a_0 + a_1 x}{1 + p_1 x + p_2 x^2} \equiv f(x). \quad (T_1)$$

Here

$$u_0 = 1, \ u_1 = 1, \ u_2 = 5, \ \cdots.$$

$$\left.\begin{array}{r} 5 + p_1 \ + p_2 = 0 \\ 13 + 5p_1 + p_2 = 0 \end{array}\right\}. \quad \therefore p_1 = -2, \ p_2 = -3.$$

Substituting in the second member of (T_1), clearing of fractions, and finding the values of a_0 and a_1, we find

$$f(x) \equiv \frac{1 - x}{1 - 2x - 3x^2}.$$

85. The coefficients of the quotient Q of two polynomials P_1 and P_2, and the coefficients of the remainder R, can always be expressed as determinants in terms of the coefficients of P_1 and P_2.

The method employed in the following example is applicable in general.

Let

$$\begin{aligned} P_1 &\equiv a_0 x^5 + a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5; \\ P_2 &\equiv b_0 x^3 + b_1 x^2 + b_2 x + b_3; \\ Q &\equiv q_0 x^2 + q_1 x + q_2; \\ R &\equiv r_0 x^2 + r_1 x + r_2. \end{aligned}$$

Now

$$P_2 Q + R \equiv P_1;$$

hence

$$(p) \left\{ \begin{aligned} a_0 &= b_0 q_0, \\ a_1 &= b_1 q_0 + b_0 q_1, \\ a_2 &= b_2 q_0 + b_1 q_1 + b_0 q_2, \\ a_3 &= b_3 q_0 + b_2 q_1 + b_1 q_2 + r_0, \\ a_4 &= \phantom{b_3 q_0 + {}} b_3 q_1 + b_2 q_2 + r_1, \\ a_5 &= \phantom{b_3 q_0 + b_2 q_1 + {}} b_3 q_2 + r_2. \end{aligned} \right.$$

From the first three of equations (p) we can find q_0, q_1, q_2, and then taking the first three with each of the others in succession, we obtain r_0, r_1, r_2. For example,

$$b_0^3 q_2 = \begin{vmatrix} b_0 & 0 & a_0 \\ b_1 & b_0 & a_1 \\ b_2 & b_1 & a_2 \end{vmatrix}; \quad b_0^3 r_0 = \begin{vmatrix} b_0 & 0 & 0 & a_0 \\ b_1 & b_0 & 0 & a_1 \\ b_2 & b_1 & b_0 & a_2 \\ b_3 & b_2 & b_1 & a_3 \end{vmatrix}.$$

Let the student find the remaining coefficients.

86. The coefficients of any equation can be expressed in terms of the roots as the quotient of two determinants, as follows. The method employed is applicable in general. By reference to examples 6 and 7, page 37, it is readily seen that if

$$f(x) \equiv x^3 - a_1 x^2 + a_2 x - a_3 \equiv (x-\alpha)(x-\beta)(x-\gamma),$$

we have

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x & \alpha & \beta & \gamma \\ x^2 & \alpha^2 & \beta^2 & \gamma^2 \\ x^3 & \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} \equiv -(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(x-\alpha)(x-\beta)(x-\gamma).$$

Expanding the first member,

$$-\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} + x \begin{vmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} - x^2 \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} + x^3 \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$$

$$\equiv \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} (x^3 - a_1 x^2 + a_2 x - a_3).$$

From this identity the required expressions in determinant form are at once obtained by equating the coefficients of like powers of x.

87. With the aid of determinants we readily find the sum of the like powers of the roots of any equation, as follows:

Let $s_1, s_2, s_3 \dots s_n$ denote as usual the sum of the first, second, ... nth powers of the roots of

$$x^m + p_1 x^{m-1} + p_2 x^{m-2} + \cdots + p_{m-1} x + p_m = 0. \tag{1}$$

Then from the theory of equations we have

$$\left.\begin{array}{llllll}
p_1 & +s_1 & & & & =0 \\
2p_2 & +p_1 s_1 & + \ s_2 & & & =0 \\
3p_3 & +p_2 s_1 & + \ p_1 s_2 & + \ s_3 & & =0 \\
\cdots & \cdots & \cdots & \cdots & & \cdots \\
\multicolumn{5}{l}{(n-1)p_{n-1} + p_{n-2}s_1 + p_{n-3}s_2 + p_{n-4}s_3 + \cdots + s_{n-1}} & =0 \\
np_n & \multicolumn{4}{l}{+p_{n-1}s_1 + p_{n-2}s_2 + p_{n-3}s_3 + \cdots + p_1 s_{n-1} + s_n} & = 0
\end{array}\right\}. \tag{S}$$

From equations (S) we obtain at once

$$s_n = (-1)^n \begin{vmatrix} p_1 & 1 & 0 & \cdots & 0 & 0 \\ 2p_2 & p_1 & 1 & \cdots & 0 & 0 \\ 3p_3 & p_2 & p_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (n-1)p_{n-1} & p_{n-2} & p_{n-3} & \cdots & p_1 & 1 \\ np_n & p_{n-1} & p_{n-2} & \cdots & p_2 & p_1 \end{vmatrix}.$$

If in (1) the coefficient of x^n had been p_0, we should, of course, have to write in the value of s_n just obtained, $\left(\frac{-1}{p_0}\right)^n$ instead of $(-1)^n$, and p_0 instead of 1, for each element of the minor diagonal of the determinant. If $n = 3$, and $n = 4$, the above formula gives

$$s_3 = -\begin{vmatrix} p_1 & 1 & 0 \\ 2p_2 & p_1 & 1 \\ 3p_3 & p_2 & p_1 \end{vmatrix}, \text{ and } s_4 = \begin{vmatrix} p_1 & 1 & 0 & 0 \\ 2p_2 & p_1 & 1 & 0 \\ 3p_3 & p_2 & p_1 & 1 \\ 4p_4 & p_3 & p_2 & p_1 \end{vmatrix},$$

respectively.

88. Equations (S) can also be employed to give the value of the coefficients in terms of $s_1, s_2, s_3 \dots s_n$, by solving these equations for the coefficients. We find

$$p_n = \frac{(-1)^n}{n!}\begin{vmatrix} s_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ s_2 & s_1 & 2 & 0 & \cdots & 0 & 0 \\ s_3 & s_2 & s_1 & 3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_1 & n-1 \\ s_n & s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_2 & s_1 \end{vmatrix}.$$

If, as before, the coefficient of x^n in equation (1) had been p_0, we would write in this value of p_n, $\left(\frac{-p_0}{n!}\right)^n$ instead of $\left(\frac{-1}{n!}\right)^n$. If $n=3$, and $n=4$,

$$p_3 = -\frac{1}{6}\begin{vmatrix} s_1 & 1 & 0 \\ s_2 & s_1 & 1 \\ s_3 & s_2 & s_1 \end{vmatrix}; \quad p_4 = \frac{1}{24}\begin{vmatrix} s_1 & 1 & 0 & 0 \\ s_2 & s_1 & 2 & 0 \\ s_3 & s_2 & s_1 & 3 \\ s_4 & s_3 & s_2 & s_1 \end{vmatrix}.$$

89. Any differential equation of the form

$$y_3 + X_1 y_2 + X_2 y_1 + X_3 y = 0, \qquad (1)$$

in which y, y_1, y_2, y_3 denote a function of x and its successive derivatives respectively, and X_1, X_2, X_3 are also functions of x, can be reduced to an equation of the next lower order, provided a particular solution of (1) is known.

Let $y = z$ satisfy equation (1). Then

$$z_3 + X_1 z_2 + X_2 z_1 + X_3 z = 0. \qquad (2)$$

Put $$u = y_1 - \frac{z_1}{z} y, \quad v = zu.$$

Then, as above, denoting derivatives by subscripts, we have

$$\begin{aligned} -v + zy_1 - z_1 y &= 0. \\ -v_1 + zy_2 - z_2 y &= 0. \\ -v_2 + zy_3 + z_1 y_2 - z_2 y_1 - z_3 y &= 0. \end{aligned}$$

These three equations and (1) are simultaneous; hence

$$\Delta \equiv \begin{vmatrix} 1 & X_1 & X_2 & X_3 y \\ 0 & 0 & z & -v-z_1 y \\ 0 & z & 0 & -v_1-z_2 y \\ z & z_1 & -z_2 & -v_2-z_3 y \end{vmatrix} = 0.$$

Now multiply the fourth column of Δ by $\frac{z}{y}$, then add to the fourth column the first multiplied by z_3, the second multiplied by z_2, and the third multiplied by z_1, and we have

$$\begin{vmatrix} 1 & X_1 & X_2 & 0 \\ 0 & 0 & z & -v \\ 0 & z & 0 & -v_1 \\ z & z_1 & -z_2 & -v_2 \end{vmatrix} = 0;$$

or
$$v_2 z - v_1(z_1 - X_1 z) + v(z_2 + X_2 z) = 0,$$

which is a differential equation of the second order.

Resultants, or Eliminants.

90. If we have given a system of n homogeneous equations containing n variables, or, what amounts to the same thing, n non-homogeneous equations containing $n-1$ variables, it is always possible to combine these equations in such a way as to eliminate the variables and obtain an equation of relation between the coefficients of the form

$$R = 0. \tag{1}$$

R, when expressed in a rational integral form, is called the *Resultant* or *Eliminant* of the system. In **77** and **79** we pointed out the fact that the equation $R=0$ must hold between the coefficients of a system of equations if they are consistent with each other (simultaneous). In the examples of **84** we repeatedly found the resultant of given systems of equations. Among the most important problems of elimination is the following: *to find the resultant of two given equations, containing a single variable.*

We consider first

Euler's Method of Elimination.

91. I. Given

$$f(x) \equiv p_0 x^2 + p_1 x + p_2 = 0, \tag{1}$$

and

$$\phi(x) \equiv q_0 x^2 + q_1 x + q_2. \tag{2}$$

If these equations have a common root, we must have

$$\frac{f(x)}{x-r} = (a_1 x + a_2), \quad \frac{\phi(x)}{x-r} = (b_1 x + b_2),$$

in which a_1, a_2, b_1, b_2 are undetermined, since r is unknown. Then

$$(b_1 x + b_2)(p_0 x^2 + p_1 x + p_2) \equiv (a_1 x + a_2)(q_0 x^2 + q_1 x + q_2).$$

Whence the equations

$$\begin{aligned} b_1 p_0 + 0 - a_1 q_0 + 0 &= 0. \\ b_1 p_1 + b_2 p_0 - a_1 q_1 - a_2 q_0 &= 0. \\ b_1 p_2 + b_2 p_1 - a_1 q_2 - a_2 q_1 &= 0. \\ 0 + b_2 p_2 + 0 - a_2 q_2 &= 0. \end{aligned}$$

Hence, by **77**, the resultant is

$$R \equiv \begin{vmatrix} p_0 & 0 & q_0 & 0 \\ p_1 & p_0 & q_1 & q_0 \\ p_2 & p_1 & q_2 & q_1 \\ 0 & p_2 & 0 & q_2 \end{vmatrix} = 0.$$

II. In general, let

$$f(x) \equiv p_0 x^m + p_1 x^{m-1} + p_2 x^{m-2} + \cdots + p_{m-1} x + p_m = 0. \tag{1}$$

$$\phi(x) \equiv q_0 x^n + q_1 x^{n-1} + q_2 x^{n-2} + \cdots + q_{n-1} x + q_n = 0. \tag{2}$$

Let r be a common root of (1) and (2), and put

$$\frac{f(x)}{x-r} = a_1 x^{m-1} + a_2 x^{m-2} + \cdots + a_{m-1} x + a_m \equiv f_1(x),$$

$$\frac{\phi(x)}{x-r} = b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_{n-1} x + b_n \equiv \phi_1(x),$$

in which the coefficients $a_1, a_2, \cdots a_m, b_1, b_2, \cdots b_n$ are undetermined. Then

$$f_1(x)\,\phi(x) \equiv \phi_1(x) f(x). \tag{I.}$$

From the identity (I.), by the theory of indeterminate coefficients, we must have $m+n$ homogeneous equations between the $m+n$ coefficients $a_1, a_2 \cdots a_m, b_1, b_2 \cdots b_n$. Hence the determinant of the system of these $m+n$ equations must vanish if (1) and (2) have a common root, and the resultant sought is accordingly this determinant.

As an application of Euler's method, take the following example. To find the conditions that must be fulfilled when

$$f(x) \equiv p_0x^3 + p_1x^2 + p_2x + p_3 = 0, \tag{1}$$
$$\phi(x) \equiv q_0x^3 + q_1x^2 + q_2x + q_3 = 0, \tag{2}$$

have two common roots.

If (1) and (2) have two common roots, two factors of $f(x)$ must be the same as two factors of $\phi(x)$. Hence

$$(ax+b)(p_0x^3 + p_1x^2 + p_2x + p_3) \equiv (cx+d)(q_0x^3 + q_1x^2 + q_2x + q_3),$$

where a, b, c, d are indeterminate coefficients. Whence

$$\begin{aligned}
ap_0 + 0 - cq_0 + 0 &= 0.\\
ap_1 + bp_0 - cq_1 - dq_0 &= 0.\\
ap_2 + bp_1 - cq_2 - dq_1 &= 0.\\
ap_3 + bp_2 - cq_3 - dq_2 &= 0.\\
0 + bp_3 + 0 - dq_3 &= 0.
\end{aligned}$$

From every four of these five homogeneous equations we obtain a determinant of the fourth order whose vanishing expresses one of the required conditions. Hence the conditions sought are expressed by the matrical equation

$$\left\Vert \begin{matrix} p_0 & p_1 & p_2 & p_3 & 0 \\ 0 & p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 & 0 \\ 0 & q_0 & q_1 & q_2 & q_3 \end{matrix} \right\Vert = 0.$$

Sylvester's Dialytic Method of Elimination.

92. I. Given

$$p_0x^3 + p_1x^2 + p_2x + p_3 = 0, \tag{1}$$
$$q_0x^2 + q_1x + q_2 = 0. \tag{2}$$

Multiply (1) successively by x^2, x, and (2) by x^3, x^2, x. Then we have the following system of equations:

$$\begin{aligned}
p_0x^5 + p_1x^4 + p_2x^3 + p_3x^2 \qquad\qquad &= 0.\\
p_0x^4 + p_1x^3 + p_2x^2 + p_3x &= 0.\\
q_0x^5 + q_1x^4 + q_2x^3 \qquad\qquad\qquad &= 0.\\
q_0x^4 + q_1x^3 + q_2x^2 \qquad\qquad &= 0.\\
q_0x^3 + q_1x^2 + q_2x &= 0.
\end{aligned}$$

We may consider these equations linear and homogeneous with respect to x^5, x^4, x^3, x^2, x, considered as separate variables. Hence

$$R \equiv \begin{vmatrix} p_0 & p_1 & p_2 & p_3 & 0 \\ 0 & p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & 0 & 0 \\ 0 & q_0 & q_1 & q_2 & 0 \\ 0 & 0 & q_0 & q_1 & q_2 \end{vmatrix} = 0.$$

II. In general, let

$$f(x) \equiv p_0x^m + p_1x^{m-1} + \cdots + p_{m-1}x + p_m = 0, \tag{1}$$
$$\phi(x) \equiv q_0x^n + q_1x^{n-1} + \cdots + q_{n-1}x + q_n = 0. \tag{2}$$

If we multiply (1) successively by $x, x^2 \cdots x^n$, and (2) successively by $x, x^2 \cdots x^m$, we obtain a system of $m+n$ equations, linear and homogeneous, with respect to $x, x^2, x^3, \cdots x^{m+n}$ considered as separate variables. From these equations we eliminate the variables by **77** and obtain the resultant in the form of a determinant of order $m+n$.

$$R \equiv \begin{vmatrix} p_0 & p_1 & p_2 & \cdots & p_n & p_{n+1} & p_{n+2} & \cdots \\ 0 & p_0 & p_1 & \cdots & p_{n-1} & p_n & p_{n+1} & \cdots \\ 0 & 0 & p_0 & \cdots & p_{n-2} & p_{n-1} & p_n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ q_0 & q_1 & q_2 & \cdots & q_n & 0 & 0 & \cdots \\ 0 & q_0 & q_1 & \cdots & q_{n-1} & q_n & 0 & \cdots \\ 0 & 0 & q_0 & \cdots & q_{n-2} & q_{n-1} & q_n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

It is evident from the form of R that the coefficients of (1) enter R in the degree of (2), and that the coefficients of (2) enter R in the degree of (1).

Cauchy's Modification of Bezout's Method of Elimination.

93. I. Given

$$p_0x^3 + p_1x^2 + p_2x + p_3 = 0, \tag{1}$$

and

$$q_0x^3 + q_1x^2 + q_2x + q_3 = 0. \tag{2}$$

Transposing and dividing (1) by (2), we obtain successively

$$\frac{p_0}{q_0} = \frac{p_1x^2 + p_2x + p_3}{q_1x^2 + q_2x + q_3},$$

$$\frac{p_0x + p_1}{q_0x + q_1} = \frac{p_2x + p_3}{q_2x + q_3},$$

$$\frac{p_0x^2 + p_1x + p_2}{q_0x^2 + q_1x + q_2} = \frac{p_3}{q_3}.$$

Clearing these equations of fractions, we have

$$(p_0q_1 - q_0p_1)x^2 + (p_0q_2 - q_0p_2)x + (p_0q_3 - q_0p_3) = 0,$$

$$(p_0q_2 - q_0p_2)x^2 + [(p_0q_3 - q_0p_3) + (p_1q_2 - q_1p_2)]x + (p_1q_3 - q_1p_3) = 0,$$

$$(p_0q_3 - q_0p_3)x^2 + (p_1q_3 - q_1p_3)x + (p_2q_3 - q_2p_3) = 0.$$

Eliminating x^2 and x, regarded as distinct variables, from these equations by **79**, we find

$$R \equiv \begin{vmatrix} |p_0\,q_1| & |p_0\,q_2| & |p_0\,q_3| \\ |p_0\,q_2| & |p_0\,q_3| + |p_1\,q_2| & |p_1\,q_3| \\ |p_0\,q_3| & |p_1\,q_3| & |p_2\,q_3| \end{vmatrix} = 0.$$

The resultant is found by this method in the form of an axisymmetric determinant,* whose elements are easily written, as we shall show by another example. Let the given equations be

$$p_0x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0, \tag{1}$$

and

$$q_0x^4 + q_1x^3 + q_2x^2 + q_3x + q_4 = 0. \tag{2}$$

We have, as before,

$$\left.\begin{aligned} \frac{p_0}{q_0} &= \frac{p_1x^3 + p_2x^2 + p_3x + p_4}{q_1x^3 + q_2x^2 + q_3x + q_4} \\ \frac{p_0x + p_1}{q_0x + q_1} &= \frac{p_2x^2 + p_3x + p_4}{q_2x^2 + q_3x + q_4} \\ \frac{p_0x^2 + p_1x + p_2}{q_0x^2 + q_1x + q_2} &= \frac{p_3x + p_4}{q_3x + q_4} \\ \frac{p_0x^3 + p_1x^2 + p_2x + p_3}{q_0x^3 + q_1x^2 + q_2x + q_3} &= \frac{p_4}{q_4} \end{aligned}\right\}. \tag{E}$$

Clearing equations (E) of fractions, we have

$$\begin{aligned} &|p_0\,q_1|x^3 + |p_0\,q_2|x^2 + |p_0\,q_3|x + |p_0\,q_4| = 0, \\ &|p_0\,q_2|x^3 + [|p_0\,q_3| + |p_1\,q_2|]x^2 + [|p_0\,q_4| + |p_1\,q_3|]x + |p_1\,q_4| = 0, \\ &|p_0\,q_3|x^3 + [|p_0\,q_4| + |p_1\,q_3|]x^2 + [|p_1\,q_4| + |p_2\,q_3|]x + |p_2\,q_4| = 0, \\ &|p_0\,q_4|x^3 + |p_1\,q_4|x^2 + |p_2\,q_4|x + |p_3\,q_4| = 0. \end{aligned}$$

Hence, as before, the resultant is

$$R \equiv \begin{vmatrix} |p_0\,q_1| & |p_0\,q_2| & |p_0\,q_3| & |p_0\,q_4| \\ |p_0\,q_2| & |p_0\,q_3| + |p_1\,q_2| & |p_0\,q_4| + |p_1\,q_3| & |p_1\,q_4| \\ |p_0\,q_3| & |p_0\,q_4| + |p_1\,q_3| & |p_1\,q_4| + |p_2\,q_3| & |p_2\,q_4| \\ |p_0\,q_4| & |p_1\,q_4| & |p_2\,q_4| & |p_3\,q_4| \end{vmatrix} = 0.$$

* For symmetrical determinants, see 107.

To form this resultant directly from the equations, write the two symmetrical determinants

$$\begin{vmatrix} |p_0 q_1| & |p_0 q_2| & |p_0 q_3| & |p_0 q_4| \\ |p_0 q_2| & |p_0 q_3| & |p_0 q_4| & |p_1 q_4| \\ |p_0 q_3| & |p_0 q_4| & |p_1 q_4| & |p_2 q_4| \\ |p_0 q_4| & |p_1 q_4| & |p_2 q_4| & |p_3 q_4| \end{vmatrix}, \text{ and } \begin{vmatrix} |p_1 q_2| & |p_1 q_3| \\ |p_1 q_3| & |p_2 q_3| \end{vmatrix},$$

formed from the coefficients of (1) and (2) in an obvious and easy way. It is then evident that R is formed from these two determinants by adding the elements of the second to the four inner elements of the first. If the equations are of the fifth degree, the student will form the resultant in the same way from the three determinants

$$\begin{vmatrix} |p_0 q_1| & |p_0 q_2| & |p_0 q_3| & |p_0 q_4| & |p_0 q_5| \\ |p_0 q_2| & |p_0 q_3| & |p_0 q_4| & |p_0 q_5| & |p_1 q_5| \\ |p_0 q_3| & |p_0 q_4| & |p_0 q_5| & |p_1 q_5| & |p_2 q_5| \\ |p_0 q_4| & |p_0 q_5| & |p_1 q_5| & |p_2 q_5| & |p_3 q_5| \\ |p_0 q_5| & |p_1 q_5| & |p_2 q_5| & |p_3 q_5| & |p_4 q_5| \end{vmatrix}, \quad \begin{vmatrix} |p_1 q_2| & |p_1 q_3| & |p_1 q_4| \\ |p_1 q_3| & |p_1 q_4| & |p_2 q_4| \\ |p_1 q_4| & |p_2 q_4| & |p_3 q_4| \end{vmatrix},$$

$$|p_2 q_3|,$$

by adding the third to the middle element of the second, and then adding the elements of the second to the nine inner elements of the first. This process is, of course, general.

From the preceding examples we see that by Bezout's method, *the resultant of two equations, each of the nth degree, is a symmetrical determinant of the same degree whose elements are either determinants of the second order or the sum of such determinants.*

II. If the two equations are not of the same degree, suppose we have given

$$p_0x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0, \qquad (1)$$
$$q_0x^2 + q_1x + q_2 = 0. \qquad (2)$$

Multiply (2) by x^2; the equations are then

$$p_0x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0, \qquad (1_1)$$
$$q_0x^4 + q_1x^3 + q_2x^2 = 0. \qquad (2_1)$$

From (1_1) and (2_1),

$$\frac{p_0}{q_0} = \frac{p_1x^3+p_2x^2+p_3x+p_4}{q_1x^3+q_2x^2},$$

$$\frac{p_0x+p_1}{q_0x+q_1} = \frac{p_2x^2+p_3x+p_4}{q_2x^2}.$$

Clearing these equations of fractions, we have

$$|p_0\,q_1|x^3+|p_0\,q_2|x^2-q_0p_3x-q_0p_4 = 0,$$
$$|p_0\,q_2|x^3+\{|p_1\,q_2|-q_0\,p_3\}\,x^2-(q_0\,p_4+q_1\,p_3)x-q_1p_4=0.$$

With these equations consider (2) multiplied by x, and (2),

$$q_0x^3+q_1x^2+q_2x=0,$$
$$q_0x^2+q_1x\ +q_2\ =0.$$

From these four equations eliminate x^3, x^2, x, and we have

$$R \equiv \begin{vmatrix} |p_0\,q_1| & |p_0\,q_2| & q_0\,p_3 & q_0\,p_4 \\ |p_0\,q_2| & |p_1\,q_2|-q_0\,p_3 & q_0\,p_4+q_1\,p_3 & q_1\,p_4 \\ q_0 & q_1 & -q_2 & 0 \\ 0 & q_0 & -q_1 & -q^2 \end{vmatrix} = 0.$$

III. In general, let

$$f(x) \equiv p_0x^m+p_1x^{m-1}+p_2x^{m-2}+\cdots+p_{m-1}x+p_m=0, \quad (1)$$
$$\phi(x) \equiv q_0x^n+q_1x^{n-1}+q_2x^{n-2}+\cdots+q_{n-1}x+q_n=0, \quad (2)$$

in which m is greater than n. Multiply (2) by x^{m-n}; then (2) becomes

$$q_0x^m+q_1x^{m-1}+q_2x^{m-2}+\cdots+q_{n-1}x^{m-n+1}+q_nx^{m-n}. \quad (2_1)$$

From (1) and (2_1),

$$\frac{p_0}{q_0} = \frac{p_1x^{m-1}+p_2x^{m-2}+\cdots+p_{m-1}x+p_m}{q_1x^{m-1}+q_2x^{m-2}+\cdots+q_{n-1}\,x^{m-n+1}+q_nx^{m-n}},$$

$$\frac{p_0x+p_1}{q_0x+q_1} = \frac{p_2x^{m-2}+p_3x^{m-3}+\cdots+p_{m-1}x+p_m}{q_2x^{m-2}+q_3x^{m-3}+\cdots+q_{n-1}x^{m-n+1}+q_nx^{m-n}},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$\frac{p_0x^{n-1}+p_1x^{n-2}+\cdots+p_{n-2}x+p_{n-1}}{q_0x^{n-1}+q_1x^{n-2}+\cdots+q_{n-2}x+q_{n-1}} = \frac{p_nx^{m-n}+p_{n+1}x^{m-n-1}+\cdots+p_m}{q_nx^{m-n}}.$$

Clear these equations of fractions, and consider with them the following $m-n$ equations obtained from (2) by multiplying it in order by $1, x, x^2, \cdots x^{m-n-1}$,

$$\begin{array}{l}
q_0x^{m-1}+q_1x^{m-2}+q_2x^{m-3}+\cdots+q_{n-1}x^{m-n}+q_nx^{m-n-1} \qquad\qquad =0,\\
\qquad\qquad q_0x^{m-2}+q_1x^{m-3}+\cdots+q_{n-2}x^{m-n}+q_{n-1}x^{m-n-1}q_nx^{m-n-2}=0,\\
\qquad\qquad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots\\
\qquad\qquad q_0x^n \quad +q_1x^{n-1}+\cdots+q_{n-1}x \quad +q_n \qquad\qquad =0.
\end{array}$$

From these m equations the resultant is obtained by eliminating the $m-1$ successive powers of x regarded as separate variables.

The Resultant in Terms of the Roots.

94. Given

$$f \equiv p_0x^m+p_1x^{m-1}+\cdots+p_{m-1}x+p_m=0, \qquad (a)$$

$$\phi \equiv q_0x^n+q_1x^{n-1}+\cdots+q_{n-1}x+q_n=0. \qquad (b)$$

If $\alpha_1, \alpha_2, \ldots \alpha_m$ are the roots of (a), and $\beta_1, \beta_2, \ldots \beta_n$ are the roots of (b), we have, of course,

$$f=p_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_m), \qquad (a_1)$$

$$\phi=q_0(x-\beta_1)(x-\beta_2)\cdots(x-\beta_n). \qquad (b_1)$$

Now, if in $q_0x^n+q_1x^{n-1}+\cdots+q_{n-1}x+q_n$ we substitute successively $\alpha_1, \alpha_2, \ldots \alpha_m$, ϕ takes the m corresponding values, $\phi(\alpha_1), \phi(\alpha_2) \ldots \phi(\alpha_m)$. With these m values as roots we can form an equation of the mth degree in ϕ. This equation may be found as follows. Forming the resultant of

$$p_0x^m+p_1x^{m-1}+\cdots+p_{m-1}x+p_m \quad =0, \qquad (1)$$

$$q_0x^n+q_1x^{n-1}+\cdots+q_{n-1}x+q_n-\phi=0, \qquad (2)$$

by **92**, we have

$$R_1 \equiv \begin{vmatrix} p_0 & p_1 & p_2 \cdots & p_n & p_{n+1} & p_{n+2} & \cdots \\ 0 & p_0 & p_1 \cdots & p_{n-1} & p_n & p_{n+1} & \cdots \\ 0 & 0 & p_0 \cdots & p_{n-2} & p_{n-1} & p_n & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots & \cdots \\ q_0 & q_1 & q_2 \cdots & q_n - \phi & 0 & 0 & \cdots \\ 0 & q_0 & q_1 \cdots & q_{n-1} & q_n - \phi & 0 & \cdots \\ 0 & 0 & q_0 \cdots & q_{n-2} & q_{n-1} & q_n - \phi & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

This is obviously an equation of the mth degree in ϕ, whose roots are $\phi(\alpha_1)$, $\phi(\alpha_2)$, $\phi(\alpha_3)$, $\cdots \phi(\alpha_m)$. The absolute term T of this equation is the product of its m roots multiplied by a factor.

But from the determinant R_1,

$$T = (-1)^m p_0^n \phi(\alpha_1)\, \phi(\alpha_2) \cdots \phi(\alpha_m).$$

Again, since R_1 becomes identical with $(-1)^m R$ of **92**, II., when we have made ϕ vanish, we see that

$$R = p_0^n \phi(\alpha_1)\, \phi(\alpha_2) \cdots \phi(\alpha_m).$$

In just the same way we can show that

$$T' = (-1)^n q_0^m f(\beta_1)\, f(\beta_2) \cdots f(\beta_n);$$

and hence, after suitable interchanges of lines,

$$R = (-1)^{mn} q_0^m f(\beta_1) f(\beta_2) \cdots f(\beta_n).$$

95. These forms of the resultant R may be obtained by symmetric functions, as follows:

$$f(x) \equiv p_0 x^m + p_1 x^{m-1} + p_2 x^{m-2} + \cdots + p_{m-1}x + p_m = 0, \quad (a)$$
$$\phi(x) \equiv q_0 x^n + q_1 x^{n-1} + q_2 x^{n-2} + \cdots + q_{n-1}x + q_n = 0. \quad (b)$$

Then $\alpha_1, \alpha_2, \cdots \alpha_m$ being the roots of (a), and $\beta_1, \beta_2, \cdots \beta_n$ the roots of (b),

$$f(x) \equiv p_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_m),$$
$$\phi(x) \equiv q_0(x-\beta_1)(x-\beta_2)\cdots(x-\beta_n).$$

Now, if (a) and (b) have a common root, the product

$$f(\beta_1)f(\beta_2)\cdots f(\beta_n) \equiv P$$

must vanish, since in that case some one of the factors vanishes. The same statement applies to the product

$$\phi(\alpha_1)\phi(\alpha_2)\cdots\phi(\alpha_m) \equiv P_1.$$

But
$$\begin{aligned} f(\beta_1) &= p_0(\beta_1-\alpha_1)(\beta_1-\alpha_2)\cdots(\beta_1-\alpha_m),\\ f(\beta_2) &= p_0(\beta_2-\alpha_1)(\beta_2-\alpha_2)\cdots(\beta_2-\alpha_m),\\ &\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\ f(\beta_n) &= p_0(\beta_n-\alpha_1)(\beta_n-\alpha_2)\cdots(\beta_n-\alpha_m);\end{aligned}$$

also
$$\begin{aligned} \phi(\alpha_1) &= q_0(\alpha_1-\beta_1)(\alpha_1-\beta_2)\cdots(\alpha_1-\beta_n),\\ \phi(\alpha_2) &= q_0(\alpha_2-\beta_1)(\alpha_2-\beta_2)\cdots(\alpha_2-\beta_n),\\ &\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\\ \phi(\alpha_m) &= q_0(\alpha_m-\beta_1)(\alpha_m-\beta_2)\cdots(\alpha_m-\beta_n).\end{aligned}$$

P is accordingly made up of mn factors of the form $\beta_r - \alpha_s$. We may therefore write

$$P = p_0^n \Pi(\beta_r - \alpha_s),$$

where r has all integral values from 1 to n, and s has all integral values from 1 to m. P is moreover a symmetric function of the roots of $\phi(x)=0$, and can therefore always be expressed as a rational integral function of the coefficients; and since it vanishes when $f(x)=0$ and $\phi(x)=0$ have a common root, and not otherwise, when P is expressed in terms of the coefficients, P is the resultant of (a) and (b). In the same way

$$P_1 = q_0^m \Pi(\alpha_s - \beta_r) = (-1)^{mn} q_0^m \Pi(\beta_r - \alpha_s),$$

where s and r have the same values as before. Hence we may write the resultant

$$R \equiv (-1)^{mn} q_0^m f(\beta_1) f(\beta_2)\cdots f(\beta_n) = p_0^n \phi(\alpha_1)\phi(\alpha_2)\cdots\phi(\alpha_m), \quad (A)$$

for both these expressions are rational integral functions of the coefficients of $f(x)$ and $\phi(x)$, which vanish when $f(x)=0$ and $\phi(x)=0$ have a common root, and not otherwise, and which become identical when expressed in terms of the coefficients. The value of R can accordingly be written

$$\pm R = p_0^n q_0^m \Pi(\beta_r - \alpha_s). \tag{B}$$

Properties of the Resultant.

96. I. By reference to the forms (A), we observe that the coefficients $p_0, p_1 \cdots p_m$ of equation (a) enter the resultant in the nth degree, and the coefficients $q_0, q_1 \cdots q_n$ of (b) enter the resultant in the mth degree; moreover, we readily see that $(-1)^{mn} q_0^m p_m^n$ is a term from the first form of the resultant, and $p_0^n q_n^m$ is a term from the second form; hence, *given two equations of degree m and n respectively, the order of the resultant R in the coefficients is m + n; the coefficients of the first equation are found in R in the degree of the second, and the coefficients of the second equation enter R in the degree of the first.*

II. If the roots of (a) and (b) are multiplied by k, R is multiplied by k^{mn}. Since each of the mn binomial factors of

$$\pm R = p_0^n q_0^m \Pi(\beta_r - \alpha_s)$$

is in this case multiplied by k, the truth of the statement is obvious. This result is frequently expressed by saying *the weight of the resultant is mn.**

III. *If the roots of (a) and (b) are increased by h, the resultant of the transformed equations is the same as the resultant of the original equations.* This, too, is obvious, for none of the factors of R is changed when both roots are increased or diminished by the same number.

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* By the *weight* of any term is meant the degree in all the quantities that enter it. The weight of ab^2c^3 is 6.

IV. If the roots of (a) and (b) are changed into their reciprocals, the resultant R_1 of the transformed equation is $(-1)^{mn}R$.

Putting $y=\frac{1}{x}$, (a) and (b) become respectively

$$f(y)=p_m y^m + p_{m-1}y^{m-1} + p_{m-2}y^{m-2} + \cdots + p_1 y + p_0 = 0, \quad (a_1)$$

$$\phi(y)=q_n y^n + q_{n-1}y^{n-1} + q_{n-2}y^{n-2} + \cdots + q_1 y + q_0 = 0. \quad (b_1)$$

Whence

$$R_1 = q_n{}^m p_m{}^n \Pi\left(\frac{1}{\beta_r} - \frac{1}{\alpha_s}\right) = \frac{q_n{}^m p_m{}^n (-1)^{mn}\Pi(\beta_r - \alpha_s)}{(\alpha_1\alpha_2\cdots\alpha_m)^n(\beta_1\beta_2\cdots\beta_n)^m}.$$

But

$$(\alpha_1\alpha_2\cdots\alpha_m) = \frac{(-1)^m p_m}{p_0};\ (\beta_1\beta_2\cdots\beta_n) = \frac{(-1)^n q_n}{q_0},$$

$$\therefore R_1 = p_0{}^n q_0{}^m (-1)^{mn}\Pi(\beta_r - \alpha_s) = (-1)^{mn}R;$$

hence *the resultant of the transformed equations is identical with the resultant of the original equations, or differs from it only in sign, according as mn is even or odd.*

97. Of all the methods of elimination given, the dialytic method is the most direct. Another advantage of this method is that it may obviously be employed to eliminate one of two unknowns from a pair of equations, as in the following example.

Given

$$p_0x^3 + p_1x^2y + p_2xy^2 + p_3y^3 = 0,$$
$$q_0x^2 + q_1xy + q_2y^2 + q_3 = 0.$$

To eliminate x we form the following equations:

$$\begin{aligned}
p_0x^4 + p_1x^3y + p_2x^2y^2 + p_3xy^3 &= 0,\\
p_0x^3 + p_1x^2y + p_2xy^2 + p_3y^3 &= 0,\\
q_0x^4 + q_1x^3y + (q_2y^2 + q_3)x^2 &= 0,\\
q_0x^3 + q_1x^2y + (q_2y^2 + q_3)x &= 0,\\
q_0x^2 + q_1xy + q_2y^2 + q_3 &= 0.
\end{aligned}$$

Whence
$$\begin{vmatrix} p_0 & p_1 y & p_2 y^2 & p_3 y^3 & 0 \\ 0 & p_0 & p_1 y & p_2 y^2 & p_3 y^3 \\ q_0 & q_1 y & q_2 y^2 + q_3 & 0 & 0 \\ 0 & q_0 & q_1 y & q_2 y^2 + q_3 & 0 \\ 0 & 0 & q_0 & q_1 y & q_2 y^2 + q_3 \end{vmatrix} = 0,$$
an equation containing only y.

98. The same method is also frequently applicable to the elimination of $n-1$ unknowns from a set of n equations, so as to obtain a final equation with but one unknown. It will afford the student a good exercise to find from the three equations

$$a_1 x^2 y + a_2 xz + a_3 = 0, \qquad (1)$$
$$yz - a_4 x = 0, \qquad (2)$$
$$a_5 xy + a_6 x + a_7 = 0, \qquad (3)$$

a final equation in y, as follows: First, eliminate x from (1) and (3), and also from (1) and (2), obtaining two new equations in y and z. From these equations eliminate z, and obtain

$$\begin{vmatrix} a_1 y^3 + a_2 a_4 y & 0 & a_3 a_4^2 \\ -(a_5 y + a_6) a_2 a_7 & a_5^2 a_3 y^2 + (a_1 a_7^2 + 2a_3 a_5 a_6) y + a_3 a_6^2 & 0 \\ 0 & -(a_5 y + a_6) a_2 a_7 & a_5^2 a_3 y^2 + (a_1 a_7^2 + 2a_3 a_5 a_6) y + a_3 a_6^2 \end{vmatrix} = 0,$$

an equation in y of the seventh degree.

99. A further interesting application is found in the following examples, in which three variables are eliminated from as many equations. Given

$$x_1 + x_2 + x_3 = 0, \qquad x_1^2 = a, \qquad x_2^2 = b, \qquad x_3^2 = c.$$

Multiplying the first equation successively by

$$x_1, \; x_2, \; x_3, \; x_1 x_2 x_3,$$

and substituting from the last three, we get

$$\begin{aligned} a + x_1 x_2 + x_1 x_3 \qquad\qquad &= 0, \\ b + x_1 x_2 \qquad\qquad + x_2 x_3 &= 0, \\ c \qquad\quad + x_1 x_3 + x_2 x_3 &= 0, \\ c x_1 x_2 + b x_1 x_3 + a x_2 x_3 &= 0. \end{aligned}$$

Eliminating $x_1x_2,\ x_1x_3,\ x_2x_3,$

$$\begin{vmatrix} a & 1 & 1 & 0 \\ b & 1 & 0 & 1 \\ c & 0 & 1 & 1 \\ 0 & c & b & a \end{vmatrix} = 0.$$

Had we multiplied the first equation successively by

$$1, \quad x_2x_3, \quad x_1x_3, \quad x_1x_2,$$

we should find by eliminating $x_1x_2x_3,\ x_1,\ x_2,\ x_3,$

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & c & b \\ 1 & c & 0 & a \\ 1 & b & a & 0 \end{vmatrix} = 0.$$

If the original equations are

$$x_1 + x_2 + x_3 = 0, \quad x_1^3 = a, \quad x_2^3 = b, \quad x_3^3 = c,$$

one form of the resultant is obtained by multiplying the first equation successively by

$$x_1,\ x_2,\ x_3, \quad x_2^2x_3^2, \quad x_1^2x_3^2, \quad x_1^2x_2^2, \quad x_1^2x_2x_3, \quad x_1x_2^2x_3, \quad x_1x_2x_3^2,$$

and substituting from the last three. Then by eliminating

$$x_1^2,\ x_2^2,\ x_3^2, \quad x_2x_3, \quad x_1x_3, \quad x_1x_2, \quad x_1x_2^2x_3^2, \quad x_1^2x_2x_3^2, \quad x_1^2x_2^2x_3,$$

we find

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & c & b & 0 & 0 & 0 & 1 & 0 & 0 \\ c & 0 & a & 0 & 0 & 0 & 0 & 1 & 0 \\ b & a & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & b & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & c & 1 & 1 & 0 \end{vmatrix} = 0.$$

100. For a final application of the dialytic method we select the following.

Given $$\sqrt{a_0x+a_1} + \sqrt{b_0x+b_1}+c_0=0,$$

to free the equation from radicals, we may proceed as follows.

Put $$\sqrt{a_0x+a_1}=y_1, \quad \sqrt{b_0x+b_1}=y_2.$$

Then we get at once

$$y_1+y_2+c_0=0, \tag{1}$$
$$y_1^2-a_0x-a_1=0, \tag{2}$$
$$y_2^2-b_0x-b_1=0. \tag{3}$$

From (1) and (3),

$$\begin{vmatrix} 1 & 0 & -b_0x-b_1 \\ 1 & y_1+c_0 & 0 \\ 0 & 1 & y_1+c_0 \end{vmatrix}=0. \tag{4}$$

Eliminating y_1 from (2) and (4), we have

$$\begin{vmatrix} 1 & 2c_0 & c_0^2-b_0x-b_1 & 0 \\ 0 & 1 & 2c_0 & c_0^2-b_0x-b_1 \\ 1 & 0 & -a_0x-a_1 & 0 \\ 0 & 1 & 0 & -a_0x-a_1 \end{vmatrix}=0,$$

which is the equation sought.

In general, given

$$p_1\sqrt[r_1]{f_1(x)}+p_2\sqrt[r_2]{f_2(x)}+p_3\sqrt[r_3]{f_3(x)}+\cdots+p_n\sqrt[r_n]{f_n(x)}=R,$$

in which $r_1, r_2 \cdots r_n$ are integers, and $f_1(x), f_2(x) \cdots f_n(x)$ are rational integral functions of x, we may rationalize the expression as follows. Put

$$f_1(x)=y_1^{r_1}, \quad f_2(x)=y_2^{r_2}, \quad \cdots \; f_n(x)=y_n^{r_n}.$$

Then we have a system of n equations, from which, together with

$$p_1 y_1 + p_2 y_2 + \cdots + p_n y_n = R,$$

we eliminate the n variables $y_1, y_2, \cdots y_n$, and obtain a resulting equation in x without radicals.

Discriminant of an Equation.

101. I. Given

$$f(x) \equiv p_0 x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = 0, \qquad (1)$$

and the first derivatives of $f(x)$, or

$$f'(x) \equiv np_0 x^{n-1} + (n-1) p_1 x^{n-2} + (n-2) p_2 x^{n-3} + \cdots + p_{n-1}. \qquad (2)$$

Then the resultant R of $f(x)=0$ and $f'(x)=0$ is called the *discriminant* of $f(x)=0$, since, if R vanishes, $f(x)=0$ and $f'(x)=0$ have a common root, and hence $f(x)=0$ has equal roots.

Forming the resultant of (1) and (2) by **92**, we have

$$R \equiv \begin{vmatrix} p_0 & p_1 & p_2 & \cdots & p_{n-2} & p_{n-1} & p_n & 0 & 0 & 0 & \cdots \\ 0 & p_0 & p_1 & \cdots & p_{n-3} & p_{n-2} & p_{n-1} & p_n & 0 & 0 & \cdots \\ 0 & 0 & p_0 & \cdots & p_{n-4} & p_{n-3} & p_{n-2} & p_{n-1} & p_n & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ np_0 & (n-1)p_1 & (n-2)p_2 & \cdots & 2p_{n-2} & p_{n-1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & np_0 & (n-1)p_1 & \cdots & 3p_{n-3} & 2p_{n-2} & p_{n-1} & 0 & 0 & 0 & \cdots \\ 0 & 0 & np_0 & \cdots & 4p_{n-4} & 3p_{n-3} & 2p_{n-2} & p_{n-1} & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

in which the first $(n-1)$ rows are formed from the coefficients of (1), and the last n rows from the coefficients of (2).

Now multiply the first row of R by n, and subtract it from the nth row; the nth row becomes

$$0 \quad -p_1 \quad -2p_2 \cdots -(n-2)p_{n-2} \quad -(n-1)p_{n-1} \quad -np_n \quad 0 \cdots 0.$$

Hence R is at once reducible to a determinant of order $2n-2$ multiplied by p_0; calling this determinant Δ, we have

$$R = p_0 \Delta.$$

Now $\qquad R = p_0^{n-1} f'(a_1) f'(a_2) f'(a_3) \cdots f'(a_n)$; (**94** or **95**, A)

and since $\quad f'(x) = \dfrac{f(x)}{x-a_1} + \dfrac{f(x)}{x-a_2} + \dfrac{f(x)}{x-a_3} + \cdots + \dfrac{f(x)}{x-a_n},$

$$\left.\begin{array}{ll}
f'(a_1) & = p_0(a_1-a_2)(a_1-a_3) \cdots (a_1-a_{n-1})(a_1-a_n) \\
f'(a_2) & = p_0(a_2-a_1)(a_2-a_3) \cdots (a_2-a_{n-1})(a_2-a_n) \\
\cdots & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
f'(a_{n-1}) & = p_0(a_{n-1}-a_1)(a_{n-1}-a_2) \cdots (a_{n-1}-a_{n-2})(a_{n-1}-a_n) \\
f'(a_n) & = p_0(a_n-a_1)(a_n-a_2) \cdots (a_n-a_{n-2})(a_n-a_{n-1})
\end{array}\right\} . \quad (E)$$

If we multiply equations (E) together, we see that the second member of the result will contain the product of the squares of the differences of the roots $a_1, a_2, \ldots a_n$ of (1). Employing the usual notation for this product, viz., $\zeta(a_1, a_2, a_3, \cdots a_n)$, we have

$$f'(a_1) f'(a_2) \cdots f'(a_n) = (-1)^{\frac{n}{2}(n-1)} p_0^n \zeta(a_1, a_2, a_3, \cdots a_n);$$

$$\therefore \Delta = (-1)^{\frac{n}{2}(n-1)} p_0^{2n-2} \zeta(a_1, a_2, a_3, \cdots a_n).$$

II. The discriminant of an equation can also be obtained as follows:

$$f(x) = 0, \ (1) \qquad \text{and} \qquad f'(x) = 0; \qquad (2)$$

being simultaneous equations when $f(x)=0$ has equal roots, the equation

$$nf(x) - xf'(x) = 0 \qquad (3)$$

is also consistent with (1) and (2). Now (3) is an equation of the $(n-1)$th degree; and finding the resultant of (3) and $f'(x)=0$, which is also of the $(n-1)$th degree, we obtain the discriminant Δ as a determinant of order $2n-2$. For an example, we shall find the discriminant of the cubic

$$p_0 x^3 + p_1 x^2 + p_2 x + p_3 = 0.$$

We have to find the resultant Δ of the equations

$$p_1x^2 + 2p_2x + 3p_3 = 0,$$
$$3p_0x^2 + 2p_1x + p_2 = 0.$$

$$\Delta = \begin{vmatrix} p_1 & 2p_2 & 3p_3 & 0 \\ 0 & p_1 & 2p_2 & 3p_3 \\ 3p_0 & 2p_1 & p_2 & 0 \\ 0 & 3p_0 & 2p_1 & p_2 \end{vmatrix} = 0.$$

By the same process we find the discriminant of the biquadratic

$$P \equiv p_0x^4 + 4p_1x^3 + 6p_2x^2 + 4p_3x + p_4 = 0\ *$$

to be

$$\Delta \equiv \begin{vmatrix} p_0 & 3p_1 & 3p_2 & p_3 & 0 & 0 \\ 0 & p_0 & 3p_1 & 3p_2 & p_3 & 0 \\ 0 & 0 & p_0 & 3p_1 & 3p_2 & p_3 \\ p_1 & 3p_2 & 3p_3 & p_4 & 0 & 0 \\ 0 & p_1 & 3p_2 & 3p_3 & p_4 & 0 \\ 0 & 0 & p_1 & 3p_2 & 3p_3 & p_4 \end{vmatrix} = 0.$$

This is accordingly the same as $I^3 - 27J^2 = 0$, where

$$I \equiv p_0p_4 - 4p_1p_3 + 3p_2^2,$$
$$J \equiv p_0p_2p_4 + 2p_1p_2p_3 - p_0p_3^2 - p_1^2p_4 - p_2^3.$$

102. We may show that $J = 0$ is one of the necessary conditions when the biquadratic $P = 0$ of the preceding article has three equal roots. Since

$$P \equiv p_0x^4 + 4p_1x^3 + 6p_2x^2 + 4p_3x + p_4 = 0 \qquad (1)$$

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* In many processes it is found more convenient to write a given function in the form of this equation, *i.e.*,

$$p_0x^n + np_1x^{n-1} + \frac{n}{2\,!}(n-1)p_2x^{n-2} + \frac{n}{3\,!}(n-1)(n-2)p_3x^{n-3} + \cdots$$
$$+ \frac{n}{2\,!}(n-1)p_{n-2}x^2 + np_{n-1}x + p_n,$$

in which each term is multiplied by the corresponding coefficient in the expansion of $(x+1)^n$. Any given polynomial can, of course, be at once reduced to this form.

has three equal roots, two of these will be roots of

$$p_0x^3+3p_1x^2+3p_2x+p_3=0, \tag{2}$$

and one of them is a root of

$$p_0x^2+2p_1x+p_2=0. \tag{3}$$

From (2) and (3) this root is also found in

$$p_1x^2+2p_2x+p_3=0. \tag{4}$$

Multiplying (3) by x^2, (4) by $2x$, and adding, we obtain

$$x^2(p_0x^2+2p_1x+p_2)+2x(p_1x^2+2p_2x+p_3)=0. \tag{5}$$

Now adding $p_2x^2+2p_3x+p_4$ to the first member of (5), we have, since $P=0$,

$$x^2(p_0x^2+2p_1x+p_2)+2x(p_1x^2+2p_2x+p_3)+p_2x^2+2p_3x+p_4=0.$$

Hence, if (1) has three equal roots,

$$\begin{aligned} p_0x^2+2p_1x+p_2&=0,\\ p_1x^2+2p_2x+p_3&=0,\\ p_2x^2+2p_3x+p_4&=0.\end{aligned} \qquad \therefore \begin{vmatrix} p_0 & p_1 & p_2\\ p_1 & p_2 & p_3\\ p_2 & p_3 & p_4\end{vmatrix}=0,$$

or

$$J=0.$$

The other condition for three equal roots of (1) is accordingly $I=0$.

103. The resultant of a system of n homogeneous equations, one of which is of the second degree, and the remaining $n-1$ are linear, may be obtained as follows. Given

$$P\equiv p_0x^2+p_1y^2+p_2z^2+2q_0xy+2q_1xz+2q_2yz=0, \tag{1}$$

$$P_1\equiv a_1x+b_1y+c_1z=0, \tag{2}$$

$$P_2\equiv a_2x+b_2y+c_2z=0. \tag{3}$$

Differentiating (1) with respect to x, y, z in succession, and remembering Euler's theorem on homogeneous functions, we obtain

$$\begin{aligned} P=x(p_0x+q_0y+q_1z)&+y(q_0x+p_1y+q_2z)\\ &+z(q_1x+q_2y+p_2z)=0. \end{aligned} \tag{4}$$

Equations (2) and (3) and (4) are simultaneous homogeneous equations; hence, by **77**, (4) must be expressible linearly in terms of (2) and (3), and

$$\theta_1 P_1 + \theta_2 P_2 = 0 \tag{5}$$

is an equation identical with (4). Equating the coefficients of (4) and (5), we have the following system of equations:

$$\left.\begin{aligned} p_0 x + q_0 y + q_1 z - \theta_1 a_1 - \theta_2 a_2 = 0, \\ q_0 x + p_1 y + q_2 z - \theta_1 b_1 - \theta_2 b_2 = 0, \\ q_1 x + q_2 y + p_2 z - \theta_1 c_1 - \theta_2 c_2 = 0. \end{aligned}\right\} \tag{E}$$

Now, taking equations (2) and (3) with equations (E), we have a system of five homogeneous equations. Eliminating x, y, z, θ_1, θ_2, the resultant of (1), (2), (3) is

$$R \equiv \begin{vmatrix} p_0 & q_0 & q_1 & a_1 & a_2 \\ q_0 & p_1 & q_2 & b_1 & b_2 \\ q_1 & q_2 & p_2 & c_1 & c_2 \\ a_1 & b_1 & c_1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 \end{vmatrix}.$$

In general, let the system of equations be

$$\begin{aligned} f(x) \equiv p_1 x_1^2 + p_2 x_2^2 + p_3 x_3^2 + \cdots + p_n x_n^2 + 2 q_1 x_1 x_2 \\ + 2 q_2 x_1 x_3 + \cdots + 2 q_{\binom{n}{2}} x_{n-1} x_n = 0, \end{aligned}$$

$$\left.\begin{array}{l} P_1 \equiv a_1 x_1 + b_1 x_2 + c_1 x_3 + \cdots + l_1 x_n = 0 \\ P_2 \equiv a_2 x_1 + b_2 x_2 + c_2 x_3 + \cdots + l_2 x_n = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ P_{n-1} \equiv a_{n-1} x_1 + b_{n-1} x_2 + c_{n-1} x_3 + \cdots + l_{n-1} x_n = 0 \end{array}\right\}. \tag{a}$$

We have, as before, if f_{x_i}' denote the differential coefficient of $f(x)$ with respect to x_i,

$$x_1 f_{x_1}' + x_2 f_{x_2}' + x_3 f_{x_3}' + \cdots + x_n f_{x_n}' = 2 f(x) = 0. \tag{b}$$

Since (a) and (b) constitute a system of simultaneous homogeneous equations, (b) considered linear with respect to

the variables, must be expressible linearly in terms of the $n-1$ linear equations of (a). Hence (b) is identical with

$$\theta_1 P_1 + \theta_2 P_2 + \theta_3 P_3 + \cdots + \theta_{n-1} P_{n-1} = 0. \qquad (c)$$

Equating the coefficients of (b) and (c), we obtain the n homogeneous equations

$$\begin{array}{l}
p_1 x_1 + q_1 x_2 + q_2 x_3 + \cdots + q_{n-1} x_n = a_1\theta_1 + a_2\theta_2 + a_3\theta_3 + \cdots + a_{n-1}\theta_{n-1}, \\
q_1 x_1 + p_2 x_2 + q_n x_3 + \cdots + q_{2n-1} x_n = b_1\theta_1 + b_2\theta_2 + b_3\theta_3 + \cdots + b_{n-1}\theta_{n-1}, \\
q_2 x_1 + q_n x_2 + p_3 x_3 + \cdots + q_{3n-3} x_3 = c_1\theta_1 + c_2\theta_2 + c_3\theta_3 + \cdots + c_{n-1}\theta_{n-1}, \\
\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
q_{n-1} x_1 + q_{2n-1} x_2 + q_{3n-3} x_3 + \cdots + p_n x_n = l_1\theta_1 + l_2\theta_2 + l_3\theta_3 + \cdots + l_{n-1}\theta_{n-1}.
\end{array}$$

These equations, together with the $n-1$ linear equations of (a), form a system of $2n-1$ equations between $x_1, x_2, \cdots x_n$, $\theta_1, \theta_2, \cdots \theta_{n-1}$. Hence the resultant of the given system is

$$\begin{vmatrix}
p_1 & q_1 & q_2 & \cdots & q_{n-1} & a_1 & a_2 & \cdots & a_{n-1} \\
q_1 & p_2 & q_n & \cdots & q_{2n-1} & b_1 & b_2 & \cdots & b_{n-1} \\
q_2 & q_n & p_3 & \cdots & q_{3n-3} & c_1 & c_2 & \cdots & c_{n-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
q_{n-1} & q_{2n-1} & q_{3n-3} & \cdots & p_n & l_1 & l_2 & \cdots & l_{n-1} \\
a_1 & b_1 & c_1 & \cdots & l_1 & 0 & 0 & \cdots & 0 \\
a_2 & b_2 & c_2 & \cdots & l_2 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{n-1} & b_{n-1} & c_{n-1} & \cdots & l_{n-1} & 0 & 0 & \cdots & 0
\end{vmatrix}.$$

Special Solutions of Simultaneous Quadratics.

104. By the help of a special expedient we may often solve a pair of simultaneous quadratics much more rapidly and elegantly with determinants than by the ordinary methods. The following examples will serve to exemplify the method employed, and are, moreover, such forms as occur frequently.

A. Find x and y in

$$\left.\begin{array}{c} \dfrac{a_1 x + b_1 y}{a_2 x + b_2 y} = \dfrac{m_1}{m_2} \\ x^2 + y^2 = r^2 \end{array}\right\}. \qquad (1)$$

Let f be such a factor that

$$\left.\begin{aligned} a_1x+b_1y=fm_1 \\ a_2x+b_2y=fm_2 \end{aligned}\right\}. \qquad (2)$$

From (2)

$$x=\frac{f\begin{vmatrix} m_1 & b_1 \\ m_2 & b_2 \end{vmatrix}}{|a_1 \quad b_2|}\equiv\frac{fD}{\Delta}; \qquad y=\frac{f\begin{vmatrix} a_1 & m_1 \\ a_2 & m_2 \end{vmatrix}}{|a_1 \quad b_2|}\equiv\frac{fD_1}{\Delta}.$$

Substituting in the second equation of (1)

$$f^2D^2+f^2D_1^2=r^2\Delta^2; \qquad \therefore\ f=\frac{r\Delta}{\pm\sqrt{D^2+D_1^2}}.$$

$$\therefore\ x=\frac{rD}{\pm\sqrt{D^2+D_1^2}}; \qquad y=\frac{rD_1}{\pm\sqrt{D^2+D_1^2}}.$$

B. Solve the equations

$$\left.\begin{aligned} a_1x+b_1y=m_1xy \\ a_2x+b_2y=m_2xy \end{aligned}\right\}. \qquad (1)$$

Divide these equations member by member; then, as before, put

$$\left.\begin{aligned} a_1x+b_1y=fm_1 \\ a_2x+b_2y=fm_2 \end{aligned}\right\}. \qquad (2)$$

$$\therefore\ x=\frac{f\,|m_1 \quad b_2|}{|a_1 \quad b_2|}; \qquad y=\frac{f\,|a_1 \quad m_2|}{|a_1 \quad b_2|}.$$

From the first equation of (1)

$$f=\frac{[a_1\,|m_1 \quad b_2|+b_1\,|a_1 \quad m_2|]\ |a_1 \quad b_2|}{m_1\,|m_1 \quad b_2|\,|a_1 \quad m_2|}.$$

$$\therefore\ x=\frac{|a_1 \quad b_2|}{|a_1 \quad m_2|}; \qquad y=\frac{|a_1 \quad b_2|}{|m_1 \quad b_2|}.$$

A shorter solution is obtained by dividing each equation of (1) by xy, and solving for $\frac{1}{x}$ and $\frac{1}{y}$.

C. Solve the equations

$$\left.\begin{aligned} a_1x + b_1y &= m_1 \\ a_2x^2 + b_2y^2 &= m_2 \end{aligned}\right\}. \qquad (1)$$

Write these equations

$$\left.\begin{aligned} a_1x + b_1y &= m_1 \\ a_2x\cdot x + b_2y\cdot y &= m_2 \end{aligned}\right\}. \qquad (2)$$

Then $x = \dfrac{\begin{vmatrix} m_1 & b_1 \\ m_2 & b_2y \end{vmatrix}}{\Delta}$; $\quad y = \dfrac{\begin{vmatrix} a_1 & m_1 \\ a_2x & m_2 \end{vmatrix}}{\Delta}$. $\quad \left(\Delta \equiv \begin{vmatrix} a_1 & b_1 \\ a_2x & b_2y \end{vmatrix}.\right)$

We have

$$\begin{aligned} x\Delta - m_1b_2y &= -m_2b_1, \\ m_1a_2x + \Delta y &= a_1m_2, \\ a_2b_1x - a_1b_2y &= -\Delta. \end{aligned}$$

Hence

$$\begin{vmatrix} \Delta & -m_1b_2 & m_2b_1 \\ m_1a_2 & \Delta & -a_1m_2 \\ a_2b_1 & -a_1b_2 & \Delta \end{vmatrix} = 0.$$

From which

$$\Delta = \pm\sqrt{a_1^2b_2m_2 + b_1^2a_2m_2 - m_1^2a_2b_2}.$$

Again,

$$\begin{aligned} a_1x + b_1y &= m_1, \\ a_2b_1x - a_1b_2y &= -\Delta. \end{aligned}$$

$\therefore\ x = -\dfrac{\begin{vmatrix} m_1 & b_1 \\ \Delta & a_1b_2 \end{vmatrix}}{\Delta_1}$; $\quad y = \dfrac{\begin{vmatrix} a_1 & m_1 \\ a_2b_1 & -\Delta \end{vmatrix}}{\Delta_1}$. $\quad \left(\Delta_1 \equiv \begin{vmatrix} a_1 & b_1 \\ a_2b_1 & -a_1b_2 \end{vmatrix}.\right)$

D. Solve the equations

$$\left.\begin{aligned} a_1x + b_1y &= m_1 \\ a_2x + b_2y + c_2xy &= m_2 \end{aligned}\right\}. \qquad (1)$$

These equations we write

$$\left.\begin{aligned} a_1x + b_1y &= m_1 \\ (a_2 + c_2y)x + b_2y &= m_2 \end{aligned}\right\}. \qquad (2)$$

$$\therefore x = \frac{|m_1 \ \ b_2|}{\Delta}; \qquad y = \frac{\begin{vmatrix} a_1 & m_1 \\ a_2 + c_2 y & m_2 \end{vmatrix}}{\Delta}. \qquad \left(\Delta \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 + c_2 y & b_2 \end{vmatrix}.\right)$$

As before,

$$\begin{aligned} (\Delta + m_1 c_2) y - a_1 m_2 + \quad m_1 a_2 &= 0, \\ - b_1 c_2 y \quad + a_1 b_2 - a_2 b_1 - \Delta &= 0. \end{aligned}$$

Whence

$$\begin{vmatrix} \Delta + m_1 c_2 & m_1 a_2 - a_1 m_2 \\ - b_1 c_2 & a_1 b_2 - a_2 b_1 - \Delta \end{vmatrix} = 0,$$

a quadratic from which Δ is found.

$$\therefore x = \frac{|m_1 \ \ b_2|}{\Delta}; \qquad y = \frac{|a_1 \ \ m_2|}{\Delta + m_1 c_2}.$$

Example B above can also be solved by the method of this example.

E. Solve the equations

$$\left.\begin{aligned} ax^2 + bxy + cy^2 &= d \\ ex^2 + fxy + gy^2 &= h \end{aligned}\right\}. \tag{1}$$

Equations (1) may be written

$$\left.\begin{aligned} x^2 + 2a_1 xy + b_1 y^2 &= m_1 \\ x^2 + 2a_2 xy + b_2 y^2 &= m_2 \end{aligned}\right\} \tag{2}$$

by easy reductions. We introduce the factor 2 for convenience in calculation. A solution analogous to D could be given. Whatever the coefficient of xy, it can, of course, be at once reduced to the form $2a_1$. We write equations (2)

$$\left.\begin{aligned} x(x + a_1 y) + y(a_1 x + b_1 y) &= m_1 \\ x(x + a_2 y) + y(a_2 x + b_2 y) &= m_2 \end{aligned}\right\}. \tag{3}$$

Then

$$x = \frac{\begin{vmatrix} m_1 & a_1 x + b_1 y \\ m_2 & a_2 x + b_2 y \end{vmatrix}}{\Delta}; \qquad y = \frac{\begin{vmatrix} x + a_1 y & m_1 \\ x + a_2 y & m_2 \end{vmatrix}}{\Delta};$$

where

$$\Delta \equiv \begin{vmatrix} x + a_1 y & a_1 x + b_1 y \\ x + a_2 y & a_2 x + b_2 y \end{vmatrix}.$$

We have

$$[\Delta + | a_1 m_2 |] x + | b_1 m_2 | y = 0,$$
$$[m_2 - m_1] x + [| a_1 m_2 | - \Delta] y = 0.$$

Whence

$$\begin{vmatrix} \Delta + | a_1 m_2 | & | b_1 m_2 | \\ m_2 - m_1 & | a_1 m_2 | - \Delta \end{vmatrix} = 0.$$

Solving this quadratic,

$$\Delta = \pm \sqrt{| a_1 m_2 |^2 + | b_1 m_2 | (m_1 - m_2)}.$$

Now
$$x = \frac{| m_1 b_2 | y}{\Delta + | a_1 m_2 |}.$$

Substitute this value of x in the first of equations (2), and we have

$$\frac{| m_1 b_2 |^2 y^2}{(\Delta + | a_1 m_2 |)^2} + \frac{2 a_1 | m_1 b_2 | y^2}{\Delta + | a_1 m_2 |} + b_1 y^2 = m_1,$$

a pure quadratic, from which the value of y can be found at once.

105. To the solutions of the last article we add the following, in which one equation is a quadratic and the other is a cubic.

Find the values of x and y in

$$\left. \begin{aligned} \frac{x^2 + xy + y^2}{x^2 - xy + y^2} &= \frac{m_1}{m_2} \\ x^3 + y^3 &= a^3 \end{aligned} \right\}. \qquad (1)$$

From the first of equations (1)

$$\left. \begin{aligned} x(x + y) + y \cdot y &= \lambda m_1 \\ x(x - y) + y \cdot y &= \lambda m_2 \end{aligned} \right\}. \qquad (2)$$

$$\therefore\ x = \frac{\lambda \begin{vmatrix} m_1 & y \\ m_2 & y \end{vmatrix}}{\Delta}; \quad y = \frac{\lambda \begin{vmatrix} x+y & m_1 \\ x-y & m_2 \end{vmatrix}}{\Delta}. \quad \left(\Delta \equiv \begin{vmatrix} x+y & y \\ x-y & y \end{vmatrix} = 2y^2.\right)$$

We have

$$\left.\begin{aligned} x\Delta - \lambda(m_1 - m_2)y = 0 \\ \lambda x(m_1 - m_2) + [\Delta - \lambda(m_1 + m_2)]y = 0 \end{aligned}\right\}. \qquad (3)$$

Whence

$$\begin{vmatrix} \Delta & -\lambda(m_1 - m_2) \\ \lambda(m_1 - m_2) & \Delta - \lambda(m_1 + m_2) \end{vmatrix} = 0.$$

From this equation

$$\Delta = \frac{\lambda}{2}\left[m_1 + m_2 \pm \sqrt{10\,m_1 m_2 - 3\,m_1^2 - 3\,m_2^2}\right].$$

Now, since $\qquad \Delta = 2y^2,$

we have to find the value of λ in order to complete the solution.

From equations (2), and the second of equations (1),

$$\left.\begin{aligned} x + y &= \frac{a^3}{\lambda m_2} \\ xy &= \frac{\lambda}{2}(m_1 - m_2) \end{aligned}\right\}. \qquad (4)$$

From equations (4), and the first of equations (2), we get

$$\lambda = \frac{a^2}{\sqrt[3]{\frac{1}{2}(3\,m_1 m_2^2 - m_2^3)}};$$

and hence

$$y = \pm\frac{a}{2}\sqrt{\frac{(m_1 + m_2) \pm \sqrt{10\,m_1 m_2 - 3\,m_1^2 - 3\,m_2^2}}{\sqrt[3]{\frac{1}{2}(3\,m_1 m_2^2 - m_2^3)}}}.$$

x may be found from the second of equations (1), or from the first of equations (3).

Solution of the Cubic.

106. The general cubic equation

$$p_0 x^3 + p_1 x^2 + p_2 x + p_3 = 0 \tag{1}$$

is always reducible to the form

$$x^3 + q_1 x + q_2 = 0. \tag{2}$$

We are therefore only concerned with the solution of (2).

The determinant equation

$$\Delta \equiv \begin{vmatrix} x & a_1 & a_2 \\ a_2 & x & a_1 \\ a_1 & a_2 & x \end{vmatrix} = 0$$

is identical with

$$x^3 - 3a_1a_2x + a_1^3 + a_2^3 = 0. \tag{3}$$

We have

$$\Delta = \begin{vmatrix} x+a_1+a_2 & a_1 & a_2 \\ x+a_1+a_2 & x & a_1 \\ x+a_1+a_2 & a_2 & x \end{vmatrix};$$

hence $x+a_1+a_2$ is a factor of Δ.

Again, let α be one of the imaginary cube roots of unity; then the other is α^2. Substitute $a_1\alpha$, $a_2\alpha^2$ for a_1 and a_2 respectively in Δ, obtaining

$$\Delta' \equiv \begin{vmatrix} x & a_1\alpha & a_2\alpha^2 \\ a_2\alpha^2 & x & a_1\alpha \\ a_1\alpha & a_2\alpha^2 & x \end{vmatrix} = x^3 - 3a_1a_2\alpha^3x + a_2^3\alpha^6 + a_1^3\alpha^3 = \Delta,$$

since $\alpha^3 = \alpha^6 = 1$. Whence

$$\Delta = \begin{vmatrix} x+a_1\alpha+a_2\alpha^2 & a_1\alpha & a_2\alpha^2 \\ x+a_1\alpha+a_2\alpha^2 & x & a_1\alpha \\ x+a_1\alpha+a_2\alpha^2 & a_2\alpha^2 & x \end{vmatrix};$$

and hence Δ is divisible by $x + a_1\alpha + a_2\alpha^2$. By substituting $a_1\alpha^2$ and $a_2\alpha$ for a_1 and a_2 respectively in Δ, we obtain a determinant Δ'', which is shown equal to Δ in the same way as before.

Hence $a_1\alpha^2 + a_2\alpha + x$ is also a factor of Δ.

Accordingly,

$$\Delta = k\,(x + a_1 + a_2)\,(x + a_1\alpha + a_2\alpha^2)\,(x + a_1\alpha^2 + a_2\alpha), \tag{4}$$

where k is a numerical factor. Comparing the term x^3 of Δ with the term x^3 in the second member of (4), we see that $k = 1$.

$$\therefore\ x^3 - 3a_1a_2x + a_1^3 + a_2^3 = (x + a_1 + a_2)\,(x + a_1\alpha + a_2\alpha^2)\,(x + a_1\alpha^2 + a_2\alpha). \tag{5}$$

From (5) we have at once

$$x = -a_1 - a_2, \qquad -a_1\alpha - a_2\alpha^2, \qquad -a_1\alpha^2 - a_2\alpha.$$

Now applying this result to the solution of (2), we put

$$q_1 = -3a_1a_2, \qquad q_2 = a_1^3 + a_2^3;$$

whence

$$a_1 = \sqrt[3]{\frac{q_2}{2} \pm \sqrt{\frac{q_2^2}{4} + \frac{q_1^3}{27}}}, \qquad a_2 = \sqrt[3]{\frac{q_2}{2} \pm \sqrt{\frac{q_2^2}{4} + \frac{q_1^3}{27}}}.$$

Hence, finally, the roots of (2) are

$$\sqrt[3]{-\frac{q_2}{2} - \sqrt{\frac{q_2^2}{4} + \frac{q_1^3}{27}}} + \sqrt[3]{-\frac{q_2}{2} + \sqrt{\frac{q_2^2}{4} + \frac{q_1^3}{27}}},$$

$$\sqrt[3]{-\frac{q_2}{2} - \sqrt{\frac{q_2^2}{4} + \frac{q_1^3}{27}}}\;\frac{\sqrt{-3}-1}{2} + \sqrt[3]{-\frac{q_2}{2} + \sqrt{\frac{q_2^2}{4} + \frac{q_1^3}{27}}}\;\frac{\sqrt{-3}+1}{2},$$

$$\sqrt[3]{-\frac{q_2}{2} - \sqrt{\frac{q_2^2}{4} + \frac{q_1^3}{27}}}\;\frac{\sqrt{-3}+1}{2} + \sqrt[3]{-\frac{q_2}{2} + \sqrt{\frac{q_2^2}{4} + \frac{q_1^3}{27}}}\;\frac{\sqrt{-3}-1}{2}.$$

Symmetrical Determinants.

107. When we regard the square of elements that make up a determinant, it is natural to inquire what special properties, if any, the determinant possesses when we suppose the elements not all independent; in other words, what *special forms* arise when we suppose certain relationships to exist between the elements, and what are their most important properties. Among the special forms very frequently met with, especially in Geometry, are the *Symmetrical* determinants. The symmetry here referred to is first, symmetry with respect to the *diagonals*, and second, symmetry with respect to the *intersection of the diagonals*, *i.e.*, the centre of the square. Two elements, so situated that the row and column numbers of the one are the column and row numbers of the other, are called *conjugate elements.* Evidently the line joining two conjugate elements a_{rs} and a_{sr} is bisected at right angles by the principal diagonal. If in a determinant $a_{rs} = a_{sr}$, then the determinant is *axisymmetric*, or simply *symmetrical.* The definition of a symmetrical determinant is extended so as to mean symmetry with respect to the secondary diagonal also, so that a determinant is symmetrical if for each element there is an equal element so situated with respect to its equal that the line joining the two is bisected at right angles by one of the diagonals. The following are symmetrical determinants:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ b_1 & b_2 & c_2 & d_2 \\ c_1 & c_2 & c_3 & d_3 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{13} & a_{14} & a_{24} \\ a_{13} & a_{14} & a_{24} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & c_1 \\ a_3 & b_3 & b_2 & b_1 \\ a_4 & a_3 & a_2 & a_1 \end{vmatrix}.$$

108. We have already had a number of problems which gave rise to symmetrical determinants. The student may refer to the last determinant in example IV., **84**, to the first determinant of **84**, VII., to the form of the resultant obtained by Bezout's method of elimination, **93**, (I.), and to the value of

J, **102**, for illustrations of how symmetrical determinants occur in practice. Again, we have

$$|a_{11} \quad a_{22} \quad a_{33}|^2 =$$

$$\begin{vmatrix} a_{11}^2+a_{12}^2+a_{13}^2 & a_{11}a_{21}+a_{12}a_{22}+a_{13}a_{23} & a_{11}a_{31}+a_{12}a_{32}+a_{13}a_{33} \\ a_{21}a_{11}+a_{22}a_{12}+a_{23}a_{13} & a_{21}^2+a_{22}^2+a_{23}^2 & a_{21}a_{31}+a_{22}a_{32}+a_{23}a_{33} \\ a_{31}a_{11}+a_{32}a_{12}+a_{33}a_{13} & a_{31}a_{21}+a_{32}a_{22}+a_{33}a_{23} & a_{31}^2+a_{32}^2+a_{33}^2 \end{vmatrix},$$

which is obviously symmetrical. It is easy to show that *the square of any determinant is a symmetrical determinant.* Let

$$|a_{1n}|^2 \equiv |b_{1n}|\,;$$

then we have to show that $b_{rs} = b_{sr}$.

$$\begin{aligned} b_{rs} &= a_{r1}a_{s1} + a_{r2}a_{s2} + a_{r3}a_{s3} + \cdots + a_{rn}a_{sn}, \\ b_{sr} &= a_{s1}a_{r1} + a_{s2}a_{r2} + a_{s3}a_{r3} + \cdots + a_{sn}a_{rn}\,; \end{aligned}$$

whence the proposition. An obvious corollary is *that any even power of a determinant is a symmetrical determinant.*

109. It is evident that conjugate lines (a row and a column having the same number) in a symmetrical determinant are composed of the same elements in the same order. Consider now two minors M and M_1 of any determinant such that the rows and columns erased to obtain M are the columns and rows erased to obtain M_1. Then

$$M \equiv \begin{vmatrix} a_{fg} & a_{fh} & a_{fi} & \cdots \\ a_{gg} & a_{gh} & a_{gi} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \text{ and } M_1 \equiv \begin{vmatrix} a_{gf} & a_{gg} & \cdots \\ a_{hf} & a_{hg} & \cdots \\ a_{if} & a_{ig} & \cdots \end{vmatrix}.$$

Now, if the determinant is symmetrical, so that $a_{rs} = a_{sr}$, we have $M = M_1$, and, in particular, $A_{rs} = A_{sr}$; or, *in a symmetrical determinant, conjugate minors are equal.* From this it follows at once that *the reciprocal determinant is symmetrical.* Further, it is evident that minors whose diagonal lies in the principal diagonal of a symmetrical determinant (*coaxial minors*) are themselves symmetrical.

110. We may show that *the product of a symmetric determinant by the square of any determinant is a symmetric determinant*, as follows:

Let $|a_{1n}|$ be a symmetrical determinant, and put

$$|a_{1n}| \times |\alpha_{1n}| \equiv |c_{1n}|, \quad \text{and} \quad |\alpha_{1n}| \times |a_{1n}|^2 \equiv |b_{1n}|.$$

Then

$$b_{ik} = \alpha_{i1} c_{k1} + \alpha_{i2} c_{k2} + \alpha_{i3} c_{k3} + \cdots + \alpha_{in} c_{kn}. \tag{1}$$

In (1) substitute the values of $c_{k1}, c_{k2}, \cdots c_{kn}$, and we have

$$\begin{aligned} b_{ik} = & (a_{11} \alpha_{k1} + a_{12} \alpha_{k2} + a_{13} \alpha_{k3} + \cdots + a_{1n} \alpha_{kn}) \alpha_{i1} \\ & + (a_{21} \alpha_{k1} + a_{22} \alpha_{k2} + a_{23} \alpha_{k3} + \cdots + a_{2n} \alpha_{kn}) \alpha_{i2} \\ & + \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ & + (a_{n1} \alpha_{k1} + a_{n2} \alpha_{k2} + a_{n3} \alpha_{k3} + \cdots + a_{nn} \alpha_{kn}) \alpha_{in}, \\ & = (a_{11} \alpha_{i1} + a_{21} \alpha_{i2} + a_{31} \alpha_{i3} + \cdots + a_{n1} \alpha_{in}) \alpha_{k1} \\ & + (a_{12} \alpha_{i1} + a_{22} \alpha_{i2} + a_{32} \alpha_{i3} + \cdots + a_{n2} \alpha_{in}) \alpha_{k2} \\ & + \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ & + (a_{1n} \alpha_{i1} + a_{2n} \alpha_{i2} + a_{3n} \alpha_{i3} + \cdots + a_{nn} \alpha_{in}) \alpha_{kn}. \end{aligned}$$

Since $a_{ik} = a_{ki}$, this sum becomes

$$\alpha_{k1} c_{i1} + \alpha_{k2} c_{i2} + \cdots + \alpha_{kn} c_{in} = b_{ki}.$$

Whence $|b_{1n}|$ is symmetrical.

From this and **108** we see that *any power of a symmetrical determinant is a symmetrical determinant.*

111. Cauchy's theorem for the expansion of a determinant, example III., **63**, assumes a somewhat different form when the determinant is symmetrical. Thus, instead of

$$\Delta = a_{00} \Delta' - \Sigma a_{i0} a_{0k} A_{ik},$$

we have, when Δ is symmetrical,

$$\Delta = a_{00} \Delta' - \Sigma a_{i0}^2 A_{ii} - 2 \Sigma a_{i0} a_{k0} A_{ik},$$

in which, as before, i has all integral values from 1 to n, and

for ik we write the different combinations of the numbers $1, 2, \cdots n$, taken two at a time.

For example,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc - af^2 - bg^2 - ch^2 + 2fgh.$$

$$\begin{vmatrix} 0 & a & b & c \\ a & 0 & h & g \\ b & h & 0 & f \\ c & g & f & 0 \end{vmatrix} = a^2f^2 + b^2g^2 + c^2h^2 - 2abfg - 2acfh - 2bcgh$$
$$= (af + bg - ch)^2 - 4abfg.$$

112. Consider the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ b_1 & b_2 & c_2 & d_2 & e_2 \\ c_1 & c_2 & c_3 & d_3 & e_3 \\ d_1 & d_2 & d_3 & d_4 & e_4 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix},$$

and suppose that

$$a_1+b_1+c_1+d_1+e_1 = b_1+b_2+c_2+d_2+e_2 = c_1+c_2+c_3+d_3+e_3$$
$$= d_1+d_2+d_3+d_4+e_4 = e_1+e_2+e_3+e_4+e_5 = 0.$$

Then, first, $\Delta = 0$; since, if we add the elements of the other rows to the corresponding elements of the first row, the elements of this row all vanish; and, secondly, we can show that *all the first minors of* Δ *are equal.*

$$B_1 \equiv - \begin{vmatrix} b_1 & c_1 & d_1 & e_1 \\ c_2 & c_3 & d_3 & e_3 \\ d_2 & d_3 & d_4 & e_4 \\ e_2 & e_3 & e_4 & e_5 \end{vmatrix}, \text{ and } C_2 \equiv - \begin{vmatrix} a_1 & c_1 & d_1 & e_1 \\ b_1 & c_2 & d_2 & e_2 \\ d_1 & d_3 & d_4 & e_4 \\ e_1 & e_3 & e_4 & e_5 \end{vmatrix}.$$

The first, third, and fourth columns of B_1 are identical with the second, third, and fourth rows of C_2. By hypothesis the

elements of the first row of C_2 are respectively $-c_1$, $-c_3$, $-d_3$, $-e_3$; whence

$$C_2 = \begin{vmatrix} c_1 & c_3 & d_3 & e_3 \\ b_1 & c_2 & d_2 & e_2 \\ d_1 & d_3 & d_4 & e_4 \\ e_1 & e_3 & e_4 & e_5 \end{vmatrix} = B_1,$$

as was to be shown.

In general, *if in a symmetrical determinant the sum of the elements in each row is zero, the determinant vanishes, and all the first minors are equal.*

Let

$$\Delta \equiv | a_{00} a_{11} a_{22} \cdots a_{nn} |, \quad \text{with} \quad a_{rs} = a_{sr},$$

and

$$a_{i0} + a_{i1} + a_{i2} + \cdots + a_{in} = 0.$$

That Δ vanishes is obvious. Again,

$$A_{00} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k} & a_{1k+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k-1} & a_{2k} & a_{2k+1} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i-11} & a_{i-12} & \cdots & a_{i-1k-1} & a_{i-1k} & a_{i-1k+1} & \cdots & a_{i-1n} \\ a_{i1} & a_{i2} & \cdots & a_{ik-1} & a_{ik} & a_{ik+1} & \cdots & a_{in} \\ a_{i+11} & a_{i+12} & \cdots & a_{i+1k-1} & a_{i+1k} & a_{i+1k+1} & \cdots & a_{i+1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nk-1} & a_{nk} & a_{nk+1} & \cdots & a_{nn} \end{vmatrix}.$$

To the ith row of A_{00} add the remaining rows; the ith row becomes

$$-a_{01} \quad -a_{02} \quad \cdots \quad -a_{0k-1} \quad -a_{0k} \quad -a_{0k+1} \quad \cdots \quad -a_{0n}.$$

Then to the kth column of A_{00} add the remaining columns; the kth column becomes

$$-a_{10} \quad -a_{20} \quad \cdots \quad -a_{i-10} \quad a_{00} \quad -a_{i+10} \quad \cdots \quad -a_{n0}.$$

Now, making the ith row the first row, and the kth column the first column, we have

$$A_{00} =$$

$$(-1)^{i+k} \begin{vmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0k-1} & a_{0k+1} & \cdots & a_{0n} \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1k-1} & a_{1k+1} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i-10} & a_{i-11} & a_{i-12} & \cdots & a_{i-1k-1} & a_{i-1k+1} & \cdots & a_{i-1n} \\ a_{i+10} & a_{i+11} & a_{i+12} & \cdots & a_{i+1k-1} & a_{i+1k+1} & \cdots & a_{i+1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n0} & a_{n1} & a_{n2} & \cdots & a_{nk-1} & a_{nk+1} & \cdots & a_{nn} \end{vmatrix} = A_{ik},$$

which proves the theorem.

113. If x be subtracted from each element of the principal diagonal of a symmetrical determinant, we have a function of x which, equated to zero, gives an important equation. The roots of this equation are all real, which may be proved as follows. We have

$$f(x) \equiv \begin{vmatrix} a_{11}-x & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}-x & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-x \end{vmatrix} = 0. \quad a_{rs} = a_{sr} \qquad (1)$$

Then

$$f(-x) \equiv \begin{vmatrix} a_{11}+x & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}+x & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33}+x & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}+x \end{vmatrix}. \quad a_{rs} = a_{sr} \qquad (2)$$

Multiplying (1) and (2),

$$f(x)\, f(-x)$$

$$= \begin{vmatrix} p_{11}-x^2 & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22}-x^2 & p_{23} & \cdots & p_{2n} \\ p_{31} & p_{32} & p_{33}-x^2 & \cdots & p_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & p_{n3} & \cdots & p_{nn}-x^2 \end{vmatrix} = 0, \quad p_{rs} = p_{sr} \qquad (3)$$

where $\qquad p_{ik} \equiv a_{i1}a_{k1} + a_{i2}a_{k2} + \cdots + a_{in}a_{kn}.$

Expanding the determinant of (3) by **63**, I.,

$$|p_{1n}| - x^2\Sigma D_{n-1} + x^4\Sigma D_{n-2} - x^6\Sigma D_{n-3} + \cdots + (-x^2)^n = 0. \qquad (4)$$

Now D_{n-1}, D_{n-2}, D_{n-3}, $\cdots$, being coaxial minors of $|p_{1n}|$, are all sums of squares of minors of $|a_{1n}|$; for consider one of these minors

$$D_{n-2} \equiv \begin{vmatrix} p_{ff} & p_{fg} & \cdots & p_{fr} \\ p_{gf} & p_{gg} & \cdots & p_{gr} \\ \cdots & \cdots & \cdots & \cdots \\ p_{rf} & p_{rg} & \cdots & p_{rr} \end{vmatrix} p_{qt} = p_{tq}.$$

D_{n-2} may be obtained by squaring the array

$$\begin{matrix} a_{f1} & a_{f2} & a_{f3} & \cdots & a_{fn} \\ a_{g1} & a_{g2} & a_{g3} & \cdots & a_{gn} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1} & a_{r2} & a_{r3} & \cdots & a_{rn}, \end{matrix}$$

in which there are n columns and $n-2$ rows. By **58**, 1st, D_{n-2} must be the sum of products of pairs of determinants which in this case are equal; hence D_{n-2} is the sum of squares of minors of $|a_{1n}|$ of order $n-2$. Hence ΣD_{n-1}, ΣD_{n-2}, ΣD_{n-3}, $\cdots$, are all positive. The signs of the terms of (4) are therefore alternately positive and negative, and, by Descartes' Rule of Signs (4), can have no negative roots. Accordingly, $f(x) = 0$, or (1), cannot have a root of the form $\alpha\sqrt{-1}$, for then x^2 would be negative, which we have shown is impossible. Nor can (4) have a root of the form $\beta + \alpha\sqrt{-1}$; for if we write $a_{11} - \beta = a'_{11}$, $a_{22} - \beta = a'_{22}$, etc., the proof just given is applicable.

The student will find it interesting to apply the preceding proof to the particular case where $f(x)$ is of the third degree, *i.e.*,

$$f(x) \equiv \begin{vmatrix} a_{11} - x & a_{12} & a_{13} \\ a_{12} & a_{22} + x & a_{23} \\ a_{13} & a_{23} & a_{33} + x \end{vmatrix} = 0, \quad a_{rs} = a_{sr}$$

actually multiplying $f(x)$ by $f(-x)$, and expanding the result to obtain the equation in x^2, whose terms are alternately positive and negative.

114. Symmetrical determinants of the form

$$\begin{vmatrix} a_1 & b_2 & c_3 & d_4 & e_5 \\ f_6 & a_1 & b_2 & c_3 & d_4 \\ g_7 & f_6 & a_1 & b_2 & c_3 \\ h_8 & g_7 & f_6 & a_1 & b_2 \\ i_9 & h_8 & g_7 & f_6 & a_1 \end{vmatrix}, \quad \text{and} \quad \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & a_4 & \cdots & a_{n+1} \\ a_3 & a_4 & a_5 & \cdots & a_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_n & a_{n+1} & \cdots & a_{2n-2} \end{vmatrix}$$

$$\equiv P(a_1 a_2 \cdots a_{2n-2}),$$

are called *orthosymmetric* or *persymmetric*. That is to say, when each line perpendicular to either of the diagonals has all its elements alike, the determinant is persymmetric. Such a determinant can contain at most $2n-1$ distinct elements. Examples of the occurrence of orthosymmetric determinants in practice are found in **84**, VII.

115. The most important property of orthosymmetric determinants is that *the determinant remains unchanged when the first terms of the successive orders of differences of its $2n-1$ elements are substituted for the elements themselves.* Consider the following series of numbers, and form the 1st, 2d, 3d, $\cdots (2n-1)$th orders of differences by subtracting a_{k-1} from a_k throughout. Then adopting the usual notation, we have

$$\begin{array}{cccccccc} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \cdots & a_{2n-2} \\ & \Delta_1 & \Delta_{11} & \Delta_{12} & \Delta_{13} & \Delta_{14} & \cdots & \Delta_{1\,2n-3} \\ & & \Delta_2 & \Delta_{21} & \Delta_{22} & \Delta_{23} & \cdots & \Delta_{2\,2n-4} \\ & & & \Delta_3 & \Delta_{31} & \Delta_{32} & \cdots & \Delta_{3\,2n-5} \\ & & & & \Delta_4 & \Delta_{41} & \cdots & \Delta_{4\,2n-6} \\ & & & & \cdots & \cdots & \cdots & \cdots \\ & & & & & & & \Delta_{2n-2} \end{array}$$

We now show that

$$\Delta \equiv \begin{vmatrix} a_0 & a_1 & a_2 & a_3 & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ a_2 & a_3 & a_4 & a_5 & \cdots & a_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n-2} \end{vmatrix} = \begin{vmatrix} a_0 & \Delta_1 & \Delta_2 & \Delta_3 & \cdots & \Delta_{n-1} \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \cdots & \Delta_n \\ \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \cdots & \Delta_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta_{n-1} & \Delta_n & \Delta_{n+1} & \Delta_{n+2} & \cdots & \Delta_{2n-2} \end{vmatrix}.$$

If in Δ the $(n-1)$th, $(n-2)$th, $\cdots$ column be subtracted from the nth, $(n-1)$th, $(n-2)$th, $\cdots$ column respectively, we get

$$\Delta = \begin{vmatrix} a_0 & \Delta_1 & \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1\,n-2} \\ a_1 & \Delta_{11} & \Delta_{12} & \Delta_{13} & \cdots & \Delta_{1\,n-1} \\ a_2 & \Delta_{12} & \Delta_{13} & \Delta_{14} & \cdots & \Delta_{1\,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & \Delta_{1n-1} & \Delta_{1n} & \Delta_{1n+1} & \cdots & \Delta_{1\,2n-3} \end{vmatrix}.$$

Repeating the operation successively, we obtain

$$\Delta = \begin{vmatrix} a_0 & \Delta_1 & \Delta_2 & \Delta_3 & \cdots & \Delta_{n-1} \\ a_1 & \Delta_{11} & \Delta_{21} & \Delta_{31} & \cdots & \Delta_{n-1\,1} \\ a_2 & \Delta_{12} & \Delta_{22} & \Delta_{32} & \cdots & \Delta_{n-1\,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & \Delta_{1\,n-1} & \Delta_{2\,n-1} & \Delta_{3\,n-1} & \cdots & \Delta_{n-1\,n-1} \end{vmatrix}.$$

Operating in a similar manner upon the rows, we get

$$\Delta = \begin{vmatrix} a_0 & \Delta_1 & \Delta_2 & \Delta_3 & \cdots & \Delta_{n-1} \\ \Delta_1 & \Delta_2 & \Delta_3 & \Delta_4 & \cdots & \Delta_n \\ \Delta_2 & \Delta_3 & \Delta_4 & \Delta_5 & \cdots & \Delta_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \Delta_{n-1} & \Delta_n & \Delta_{n+1} & \Delta_{n+2} & \cdots & \Delta_{2n-2} \end{vmatrix},$$

as was to be shown.

Thus

$$\begin{vmatrix} 3 & 8 & 15 & 26 \\ 8 & 15 & 26 & 43 \\ 15 & 26 & 43 & 68 \\ 26 & 43 & 68 & 103 \end{vmatrix} = \begin{vmatrix} 3 & 5 & 2 & 2 \\ 5 & 2 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{vmatrix} = 2^4;$$

for we have

$$\begin{array}{ccccccc} 3 & 8 & 15 & 26 & 43 & 68 & 103 \\ & 5 & 7 & 11 & 17 & 25 & 35 \\ & & 2 & 4 & 6 & 8 & 10 \\ & & & 2 & 2 & 2 & 2. \end{array}$$

Similarly,

$$\begin{vmatrix} 7 & 0 & -4 & -5 \\ 0 & -4 & -5 & -3 \\ -4 & -5 & -3 & 2 \\ -5 & -3 & 2 & 10 \end{vmatrix} = \begin{vmatrix} 7 & -7 & 3 & 0 \\ -7 & 3 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0.$$

The student may show that

$$\begin{vmatrix} 1 & 2 & 4 & 7 \\ 2 & 4 & 7 & 12 \\ 4 & 7 & 12 & 19 \\ 7 & 12 & 19 & 30 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 5 \end{vmatrix}.$$

$$\begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} = 0. \qquad \begin{vmatrix} 1 & 8 & 27 & 64 \\ 8 & 27 & 64 & 125 \\ 27 & 64 & 125 & 216 \\ 64 & 125 & 216 & 343 \end{vmatrix} = 6^4.$$

Besides exhibiting obvious simplifications, these examples show that *when the elements of a persymmetric determinant of the nth degree form an arithmetical progression of order* $m^* < n - 1$, *the determinant vanishes; and if the order of the progression is* $n - 1$, *the determinant reduces to an nth power.*

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* The series of numbers

$$1 \quad 8 \quad 27 \quad 64 \quad 125 \quad 216$$

form an arithmetical progression of the third order, because the terms of the third order of differences are alike.

Thus

$$\begin{array}{cccccc} 1 & 8 & 27 & 64 & 125 & 216 \\ & 7 & 19 & 37 & 61 & 91 \\ & & 12 & 18 & 24 & 30 \\ & & & 6 & 6 & 6. \end{array}$$

116. The conditions of the last statement will always be fulfilled if a_k is a rational integral function of k of the mth degree, whose highest term has the coefficient 1. For then, according to the well-known theorem, $a_0, a_1, a_2, \cdots$ form an arithmetical series of the mth order, of which the mth differences will be $m!$. If, then, $m = n-1$, all the elements of the secondary diagonal will be $(n-1)!$, and all the elements below it will be zeros. Whence the determinant equals

$$(-1)^{\frac{n}{2}(n-1)}[(n-1)]^n.$$

If $m < n-1$, the determinant of course vanishes. In either case, instead of $a_0, a_1, a_2, \cdots$, we may write

$$a_i,\ a_{i+1},\ a_{i+2},\ \cdots.$$

If, for example, p is any given number, and

$$a_k \equiv \binom{p+k+m}{m} \equiv \frac{(p+k+m)(p+k+m-1)\cdots(p+k+1)}{m!},$$

$$\begin{vmatrix} \binom{p+m}{m} & \binom{p+m+1}{m} & \cdots & \binom{p+2m}{m} \\ \binom{p+m+1}{m} & \binom{p+m+2}{m} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \binom{p+2m}{m} & \binom{p+2m+1}{m} & \cdots & \binom{p+3m}{m} \end{vmatrix}$$

$$= (-1)^{\frac{m}{2}(m+1)} = (-1)^{(n-1)\frac{n}{2}}.$$

117. Consider the determinant

$$\Delta \equiv \begin{vmatrix} k & kr & kr^2 & \cdots & kr^{n-1} \\ kr & kr^2 & kr^3 & \cdots & kr^n \\ kr^2 & kr^3 & kr^4 & \cdots & kr^{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ kr^{n-1} & kr^n & kr^{n+1} & \cdots & kr^{2n-2} \end{vmatrix},$$

whose elements are in geometrical progression. That Δ must vanish is obvious at sight; for dividing any column except the first by the ratio r, Δ is seen to contain identical columns. *Hence if the elements of a persymmetric determinant form a geometrical progression, the determinant vanishes.*

118. To the results of the last article we add the following. Suppose in

$$\Delta \equiv \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_1 & a_2 & a_3 & \cdots & a_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_n & a_{n+1} & \cdots & a_{2n-2} \end{vmatrix}$$

each element divides every other element whose subscript is higher than its own, *i.e.*, in general,

$$a_r = b_0 b_1 b_2 \cdots b_r.$$

Then

$$\Delta = \begin{vmatrix} b_0 & b_0 b_1 & b_0 b_1 b_2 & b_0 b_1 b_2 b_3 & \cdots & b_0 b_1 b_2 \cdots b_{n-1} \\ b_0 b_1 & b_0 b_1 b_2 & b_0 b_1 b_2 b_3 & b_0 b_1 b_2 b_3 b_4 & \cdots & b_0 b_1 \cdots b_{n-1} b_n \\ b_0 b_1 b_2 & b_0 b_1 b_2 b_3 & b_0 b_1 b_2 b_3 b_4 & b_0 b_1 b_2 b_3 b_4 b_5 & \cdots & b_0 b_1 \cdots b_n b_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_0 b_1 b_2 \cdots b_{n-1} & b_0 b_1 b_2 \cdots b_n & b_0 b_1 b_2 \cdots b_{n+1} & b_0 b_1 b_2 \cdots b_{n+2} & \cdots & b_0 b_1 \cdots b_{2n-2} \end{vmatrix}.$$

Now it is obvious that b_0 is a factor of the first row of Δ, $b_0 b_1$ is a factor of the second row, $b_0 b_1 b_2$ is a factor of the third row, and so on. Hence

$$\Delta = \prod_{i=0}^{i=n-1} b_i^{n-i} \begin{vmatrix} 1 & b_1 & b_1 b_2 & b_1 b_2 b_3 & \cdots & b_1 b_2 \cdots b_{n-1} \\ 1 & b_2 & b_2 b_3 & b_2 b_3 b_4 & \cdots & b_2 b_3 \cdots b_n \\ 1 & b_3 & b_3 b_4 & b_3 b_4 b_5 & \cdots & b_3 b_4 \cdots b_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \quad \cdots \\ 1 & b_n & b_n b_{n+1} & b_n b_{n+1} b_{n+2} & \cdots & b_n b_{n+1} \cdots b_{2n-2} \end{vmatrix}.$$

Skew Determinants, and Skew Symmetrical Determinants.

119. We have heretofore shown (**108**) that the square of any determinant is a symmetrical determinant. If we now write the determinant of even order

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} -b_1 & a_1 & -d_1 & c_1 \\ -b_2 & a_2 & -d_2 & c_2 \\ -b_3 & a_3 & -d_3 & c_3 \\ -b_4 & a_4 & -d_4 & c_4 \end{vmatrix},$$

we get, by multiplying these factors together,

$$\Delta^2 =$$

$$\begin{vmatrix} 0 & -(a_1b_2)-(c_1d_2) & -(a_1b_3)-(c_1d_3) & -(a_1b_4)-(c_1d_4) \\ (a_1b_2)+(c_1d_2) & 0 & -(a_2b_3)-(c_2d_3) & -(a_2b_4)-(c_2d_4) \\ (a_1b_3)+(c_1d_3) & (a_2b_3)+(c_2d_3) & 0 & -(a_3b_4)-(c_3d_4) \\ (a_1b_4)+(c_1d_4) & (a_2b_4)+(c_2d_4) & (a_3b_4)+(c_3d_4) & 0 \end{vmatrix}.$$

In this determinant each element is equal to its conjugate with opposite sign, and the elements of the principal diagonal are zeros. Such determinants are called *skew symmetrical.* In other words, if in a determinant we have $a_{ik} = -a_{ki}$ and $a_{ii} = 0$, the determinant is *skew symmetrical.* If a_{ii} is not zero, we have a *skew* determinant. It may be shown that the square of any determinant of even order can be expressed as a skew symmetrical determinant. Thus, since

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n-3} & a_{1n-2} & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n-3} & a_{2n-2} & a_{2n-1} & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-11} & a_{n-12} & a_{n-13} & a_{n-14} & \cdots & a_{n-1n-3} & a_{n-1n-2} & a_{n-1n-1} & a_{n-1n} \\ a_{n1} & a_{n2} & a_{n3} & a_{n4} & \cdots & a_{nn-3} & a_{nn-2} & a_{nn-1} & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{12} & -a_{11} & a_{14} & -a_{13} & \cdots & a_{1n-2} & -a_{1n-3} & a_{1n} & -a_{1n-1} \\ a_{22} & -a_{21} & a_{24} & -a_{23} & \cdots & a_{2n-2} & -a_{2n-3} & a_{2n} & -a_{2n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-12} & -a_{n-11} & a_{n-14} & -a_{n-13} & \cdots & a_{n-1n-2} & -a_{n-1n-3} & a_{n-1n} & -a_{n-1n-1} \\ a_{n2} & -a_{n1} & a_{n4} & -a_{n3} & \cdots & a_{nn-2} & -a_{nn-3} & a_{nn} & -a_{nn-1} \end{vmatrix},$$

we have, after multiplying these determinants together,

$$\Delta^2 = \begin{vmatrix} 0 & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{21} & 0 & m_{23} & \cdots & m_{2n} \\ m_{31} & m_{32} & 0 & \cdots & m_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ m_{n1} & m_{n2} & m_{n3} & \cdots & 0 \end{vmatrix}_{m_{ik} = -m_{ki}}.$$

For

$$m_{ik} \equiv a_{i1} a_{k2} - a_{i2} a_{k1} + a_{i3} a_{k4} - a_{i4} a_{k3} + \cdots + a_{in-1} a_{kn} - a_{in} a_{kn-1},$$

and hence $\quad m_{ii} = 0, \quad$ and $\quad m_{ik} = -m_{ki}$.

120. The consideration of skew determinants reduces to that of skew symmetrical determinants, as we shall now show.

I. By **47**,

$$\Delta^{(n)} \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}_{a_{ik} = -a_{ki}}$$

$$= \Delta_0^{(n)} + \Sigma C_1 \Delta_0^{(n-1)} + \Sigma C_2 \Delta_0^{(n-2)} + \cdots + \Sigma C_{n-2} \Delta_0^{(2)} + C_n.$$

Now, since $a_{ik} = -a_{ki}$, the determinants $\Delta_0^{(n)}$, $\Delta_0^{(n-1)}$, $\Delta_0^{(n-2)}$, $\cdots$, are all skew symmetrical, and $\Delta^{(n)}$ is expressed in terms of skew symmetrical determinants.

II. If, further, a_{ii} in $\Delta^{(n)}$ is equal to x, we have

$$\Delta^{(n)} = \Delta_0^{(n)} + x\Sigma\Delta_0^{(n-1)} + x^2 \Sigma\Delta_0^{(n-2)} + \cdots + x^{n-2}\Delta_0^{(2)} + x^n.$$

It will soon be shown that a skew symmetrical determinant of odd order vanishes. Accordingly, the terms of this expansion in which the degree of Δ_0 is odd will vanish. Thus

$$\begin{vmatrix} x & -a & -b & -c \\ a & x & -d & -e \\ b & d & x & -f \\ c & e & f & x \end{vmatrix} = \begin{vmatrix} 0 & -a & -b & -c \\ a & 0 & -d & -e \\ b & d & 0 & -f \\ c & e & f & 0 \end{vmatrix}$$

$$+ x\left[\begin{vmatrix} 0 & -d & -e \\ d & 0 & -f \\ e & f & 0 \end{vmatrix} + \begin{vmatrix} 0 & -b & -c \\ b & 0 & -f \\ c & f & 0 \end{vmatrix} + \begin{vmatrix} 0 & -a & -c \\ a & 0 & -e \\ c & e & 0 \end{vmatrix} + \begin{vmatrix} 0 & -a & -b \\ a & 0 & -d \\ b & d & 0 \end{vmatrix}\right]$$

$$+ x^2\left[\begin{vmatrix} 0 & -f \\ f & 0 \end{vmatrix} + \begin{vmatrix} 0 & -e \\ e & 0 \end{vmatrix} + \begin{vmatrix} 0 & -d \\ d & 0 \end{vmatrix} + \begin{vmatrix} 0 & -c \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & -b \\ b & 0 \end{vmatrix} + \begin{vmatrix} 0 & -a \\ a & 0 \end{vmatrix}\right]$$

$$+ x^3[0+0+0] + x^4$$

$$= x^4 + (f^2+e^2+d^2+c^2+b^2+a^2)\,x^2 + (af-be+cd)^2.$$

The student may show that

$$\Delta \equiv \begin{vmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2.$$

Writing another skew determinant Δ_1, whose elements are e, f, g, h, in the same form as Δ just written, we see that $\Delta_1 = (e^2+f^2+g^2+h^2)^2$. If we multiply Δ and Δ_1 together by rows, we get another skew determinant Δ_2, of the same form as Δ and Δ_1; the value of Δ_2 may accordingly be written

$$(m^2+n^2+o^2+p^2)^2,$$

where

$$m \equiv ae+bf+cg+dh, \qquad o \equiv -ag+bh+ce-df,$$
$$n \equiv -af+be-ch+dg, \quad p \equiv -ah-bg+cf+de.$$

We have then

$$\Delta\Delta_1 \equiv (a^2+b^2+c^2+d^2)^2\,(e^2+f^2+g^2+h^2)^2 = (m^2+n^2+o^2+p^2)^2,$$

or $\quad (a^2+b^2+c^2+d^2)\;(e^2+f^2+g^2+h^2) = (m^2+n^2+o^2+p^2),$

which is Euler's theorem.

121. Returning now to the consideration of skew symmetrical determinants, let us take the two minors M and M_1 of **109**, and making $a_{ik} = -a_{ki}$, $a_{ii} = 0$, the determinant itself is skew symmetrical; M becomes M', and M_1 becomes M_1'. Now since every element of M' equals each element of M_1' with contrary sign, or since

$$M' \equiv \begin{vmatrix} a_{fg} & a_{fh} & a_{fi} & \cdots \\ a_{gg} & a_{gh} & a_{gi} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix}, \text{ and } M_1' \equiv \begin{vmatrix} -a_{gf} & -a_{gg} & \cdots \\ -a_{hf} & -a_{hg} & \cdots \\ -a_{if} & -a_{ig} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix},$$

$$M' = (-1)^m M_1',$$

where m, as before, is the order of the minors, *i.e.*, *the conjugate minors of a skew symmetrical determinant are equal if m is even; but if m is odd, the conjugate minors are equal, with contrary signs.*

In particular, if n is odd, $A_{ik} = A_{ki}$.

But if n is even, $A_{ik} = -A_{ki}$.

122. If the skew symmetrical determinant

$$\Delta \equiv \begin{vmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{vmatrix}$$

is multiplied by $(-1)^3$, we obtain

$$-\Delta \equiv \begin{vmatrix} 0 & -a_{12} & -a_{13} \\ a_{12} & 0 & -a_{23} \\ a_{13} & a_{23} & 0 \end{vmatrix}.$$

But since the rows of Δ are the columns of $-\Delta$,

$$\Delta = -\Delta, \quad \text{or} \quad \Delta = 0.$$

It is obvious that, in general, the effect of multiplying a skew symmetrical determinant Δ of order n by $(-1)^n$ is to change the rows into columns. Hence, when n is odd,

$$\Delta = -\Delta.$$

Therefore *a skew symmetrical determinant of odd order vanishes.* In a skew symmetrical determinant A_{ii} is, of course, skew symmetrical; hence

123. From **121**, where n is odd, the reciprocal determinant is symmetrical; and, if n is even, the reciprocal determinant is skew symmetrical.

124. I. Consider the following determinant

$$\Delta \equiv \begin{vmatrix} 0 & -a_{12} & -a_{13} & -a_{14} \\ a_{12} & 0 & -a_{23} & -a_{24} \\ a_{13} & a_{23} & 0 & -a_{34} \\ a_{14} & a_{24} & a_{34} & 0 \end{vmatrix},$$

and the reciprocal determinant

$$\Delta_1 \equiv \begin{vmatrix} 0 & A_{12} & A_{13} & A_{14} \\ -A_{12} & 0 & A_{23} & A_{24} \\ -A_{13} & -A_{23} & 0 & A_{34} \\ -A_{14} & -A_{24} & A_{34} & 0 \end{vmatrix}.$$

Now, by **61**,

$$\begin{vmatrix} 0 & A_{14} \\ -A_{14} & 0 \end{vmatrix} = \Delta \cdot \begin{vmatrix} 0 & -a_{23} \\ a_{23} & 0 \end{vmatrix}.$$

$$\therefore\ A_{14}^2 = a_{23}^2\Delta, \quad \text{or} \quad \Delta = \frac{A_{14}^2}{a_{23}^2},$$

and hence Δ is a perfect square.

II. We shall now show that, in general, *a skew symmetrical determinant of even order is a perfect square.*

Let

$$\Delta \equiv \begin{vmatrix} 0 & a_{12} & \cdots & a_{1n} & a_{1\,n+1} & \cdots & a_{1\,2n-1} & a_{1\,2n} \\ a_{21} & 0 & \cdots & a_{2n} & a_{2\,n+1} & \cdots & a_{2\,2n-1} & a_{2\,2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & 0 & a_{n\,n+1} & \cdots & a_{n\,2n-1} & a_{n\,2n} \\ a_{n+1\,1} & a_{n+1\,2} & \cdots & a_{n+1\,n} & 0 & \cdots & a_{n+1\,2n-1} & a_{n+1\,2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-1\,1} & a_{2n-1\,2} & \cdots & a_{2n-1\,n} & a_{2n-1\,n+1} & \cdots & 0 & a_{2n-1\,2n} \\ a_{2n\,1} & a_{2n\,2} & \cdots & a_{2n\,n} & a_{2n\,n+1} & \cdots & a_{2n\,2n-1} & 0 \end{vmatrix} a_{ik} = -a_{ki}.$$

Then, as above,

$$\begin{vmatrix} \Delta_{a_{11}} & \Delta_{a_{1\,2n}} \\ \Delta_{a_{2n\,1}} & \Delta_{a_{2n\,2n}} \end{vmatrix} = \Delta \cdot \Delta_{a_{11},\, a_{2n2n}}. \qquad (a)$$

Now since Δ is skew symmetrical, and n is even,

$$\Delta_{a_{11}} = \Delta_{a_{2n2n}} = 0; \quad \text{and} \quad \Delta_{a_{12n}} = -\Delta_{a_{2n1}}.$$

$$\therefore\ \Delta^2_{a_{2n1}} = \Delta \cdot \Delta_{a_{11},\, a_{2n2n}}; \quad \text{or} \quad \Delta = \frac{\Delta^2_{a_{2n1}}}{\Delta_{a_{11},\, a_{2n2n}}}. \qquad (b)$$

Therefore Δ is a perfect square if $\Delta_{a_{11},\, a_{2n2n}}$ is a perfect square. In other words, a skew symmetrical determinant of order $2n$ is a perfect square if one of the next lower even order is. But it is obvious that a skew symmetrical determinant of the second order is a perfect square, and we have shown above in I. that one of the fourth order is a perfect square; hence, by what we have just proved, a skew symmetrical determinant of the sixth order is a perfect square, and so on. Hence the theorem is true universally.

For a simple illustration, let us apply (b) to the following determinant:

$$\Delta \equiv \begin{vmatrix} 0 & -x & -y & -z \\ x & 0 & -t & -u \\ y & t & 0 & -v \\ z & u & v & 0 \end{vmatrix} = \frac{\begin{vmatrix} -x & -y & -z \\ 0 & -t & -u \\ t & 0 & -v \end{vmatrix}^2}{\begin{vmatrix} 0 & -t \\ t & 0 \end{vmatrix}} = (vx - uy + tz)^2.$$

As another application, we establish the following relation:

$$9\,(a_2-a_3)^2\,(a_1-a_4)^2\,(a_3-a_1)^2\,(a_2-a_4)^2\,(a_1-a_2)^2\,(a_3-a_4)^2$$
$$= [\,(a_2-a_3)^3\,(a_1-a_4)^3 + (a_3-a_1)^3\,(a_2-a_4)^3 + (a_1-a_2)^3\,(a_3-a_4)^3]^2.$$

The first expression equals (see example 7, page 37)

$$\begin{vmatrix} a_1^3 & a_1^2 & a_1 & 1 \\ a_2^3 & a_2^2 & a_2 & 1 \\ a_3^3 & a_3^2 & a_3 & 1 \\ a_4^3 & a_4^2 & a_4 & 1 \end{vmatrix} \times \begin{vmatrix} 1 & -3a_1 & 3a_1^2 & -a_1^3 \\ 1 & -3a_2 & 3a_2^2 & -a_2^3 \\ 1 & -3a_3 & 3a_3^2 & -a_3^3 \\ 1 & -3a_4 & 3a_4^2 & -a_4^3 \end{vmatrix}$$

$$=\begin{vmatrix} 0 & (a_1-a_2)^3 & (a_1-a_3)^3 & (a_1-a_4)^3 \\ (a_2-a_1)^3 & 0 & (a_2-a_3)^3 & (a_2-a_4)^3 \\ (a_3-a_1)^3 & (a_3-a_2)^3 & 0 & (a_3-a_4)^3 \\ (a_4-a_1)^3 & (a_4-a_2)^3 & (a_4-a_3)^3 & 0 \end{vmatrix}$$

$$=\begin{vmatrix} (a_1-a_2)^3 & (a_1-a_3)^3 & (a_1-a_4)^3 \\ 0 & (a_2-a_3)^3 & (a_2-a_4)^3 \\ (a_3-a_2)^3 & 0 & (a_3-a_4)^3 \end{vmatrix}^2 \div (a_2-a_3)^6$$

$$=[(a_2-a_3)^3(a_1-a_4)^3+(a_3-a_1)^3(a_2-a_4)^3+(a_1-a_2)^3(a_3-a_4)^3]^2,$$

as was to be shown.

125. The following proof of the preceding theorem has some advantages over the one just given. Let Δ be a skew symmetrical determinant of even order. Then $\Delta_{a_{11}}$ vanishes. Let β_{ik} be the complementary minor of a_{ik} in $\Delta_{a_{11}}$, and hence a second minor of Δ. By **60**,

$$\begin{vmatrix} \beta_{ii} & \beta_{ik} \\ \beta_{ki} & \beta_{kk} \end{vmatrix} = 0; \tag{1}$$

and since
$$\beta_{ik}=\beta_{ki}, \quad \beta_{ii}\beta_{kk}=\beta_{ik}^2. \tag{2}$$

Expanding Δ by Cauchy's theorem, **63**, III., in terms of the elements of the first row and first column, we have, since

$$\Delta_{a_{11}}=0,$$

$$\Delta=-\Sigma a_{1i}\,a_{k1}\,\beta_{ik}=\Sigma a_{1i}\,a_{1k}\sqrt{\beta_{ii}\beta_{kk}}, \text{ substituting from (2)}, \tag{3}$$

in which i, k have the values 2, 3, $\cdots$ $2n$. From (3) we have at once

$$\Delta \doteq [\Sigma a_{1i}\sqrt{\beta_{ii}}]^2.$$

Here Δ is expressed as the square of a linear function of the elements of the first row. This function is rational if $\sqrt{\beta_{ii}}$ is rational. But β_{ii} is a skew symmetrical determinant of order $2n-2$. Hence a skew symmetrical determinant of order $2n$ is a perfect square if one of order $2n-2$ is. But we proved (**124**. I.) that a skew symmetrical determinant of the

fourth order is a perfect square; hence, by what we have just proved, one of the sixth order is a perfect square, and so on.

126. Since

$$\Delta = [\Sigma a_{1i}\sqrt{\beta_{ii}}]^2$$
$$= [a_{12}\sqrt{\beta_{22}} + a_{13}\sqrt{\beta_{33}} + a_{14}\sqrt{\beta_{44}} + \cdots + a_{12n}\sqrt{\beta_{2n\,2n}}]^2;$$

that is, since Δ is the square of a linear function of the elements of the first row, we see that if Δ is of the fourth order, $\sqrt{\Delta}$ contains 3 terms; then, if Δ is of the sixth order, $\sqrt{\Delta}$ contains 5.3 terms, etc. In general, then, $\sqrt{\Delta}$ is the sum of

$$(2n-1)(2n-3)\cdots 5.\ 3.\ 1 \text{ terms.}$$

Every term of $\sqrt{\Delta}$ is, moreover, the product of n elements of Δ, in which no subscript is repeated. For, taking the term $a_{14}\sqrt{\beta_{44}}$, for instance, we see that it consists of terms in which neither of the subscripts $1, 4$ is repeated. But $\sqrt{\beta_{44}}$ will contain a term $a_{23}\sqrt{\gamma_{33}}$, in which, as before, $\sqrt{\gamma_{33}}$ contains none of the subscripts $1, 2, 3, 4$; and so on. Hence $\sqrt{\Delta}$ is the sum of terms of the form

$$a_{12}a_{34}a_{56}\cdots a_{2n-1\,2n},$$

in which no subscript is repeated.

If Δ is of the fourth order, for example, we have

$$\Delta^{(4)} \equiv \begin{vmatrix} 0 & a_{12} & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & a_{34} \\ a_{41} & a_{42} & a_{43} & 0 \end{vmatrix}, \text{ and } \sqrt{\Delta^{(4)}} = (a_{12}a_{34} \pm a_{13}a_{24} \pm a_{14}a_{23}).$$
$$a_{ik} = -a_{ki}$$

To determine which sign is prefixed to each term, we observe that since the interchange of two subscripts of Δ amounts to an interchange of two rows, and also of two columns, and therefore leaves Δ unchanged, $\sqrt{\Delta}$ must be a function in which the interchange of two subscripts either causes no change or simply a change in sign.

If we consider any term of $\sqrt{\Delta^{(4)}}$, as $a_{12}a_{34}$, which the interchange of the subscripts $1, 2$ transforms into $a_{21}a_{34} = -a_{12}a_{34}$, it is obvious that $\sqrt{\Delta^{(4)}}$ does change sign on interchanging two subscripts. We have then the square root equal to

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}; \tag{2}$$

for, if the second term of (2) were $+$, the interchange of 2 and 3, while changing the sign of the last term, leaves the signs of the first two unchanged.

Since $a_{ik} = -a_{ki}$, it is always possible to so interchange the subscripts that all the terms shall be positive. Thus

$$\sqrt{\Delta^{(4)}} = a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23}.$$

127. In general, we proceed as follows:

Δ being a skew symmetrical determinant of the $2n$th order, Δ contains the term

$$(-1)^n a_{12}a_{21}a_{34}a_{43}a_{56}a_{65}\cdots a_{2n-1\,2n}a_{2n\,2n-1} = (a_{12}a_{34}a_{56}\cdots a_{2n-1\,2n})^2.$$

Hence $\sqrt{\Delta}$ contains the term

$$+ a_{12}a_{34}a_{56}\cdots a_{2n-1\,2n} \equiv T.$$

The positive square root of Δ which contains T as its first term is an important function, possessing many properties analogous to the properties of determinants, and is called a *Pfaffian*. The notation

$$P \equiv [1, 2, \cdots 2n], \quad \text{or} \quad (1, 2, 3, \cdots 2n),$$

has been adopted for the Pfaffian. From what precedes, we see that the terms of the Pfaffian are obtained from the principal term by permuting the subscripts $2, 3, \cdots 2n$ in all possible ways, and changing sign with every permutation.

Since $a_{ik} = -a_{ki}$, we may so arrange the elements that every term of P is positive. Thus in the case of $\Delta^{(4)}$ above we have

$$\sqrt{\Delta^{(4)}} = P \equiv a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23}. \tag{p}$$

128. If two subscripts are interchanged, the sign of P is changed. Let $a_{rs}\beta$ be the terms of P containing the element a_{rs}. Then the elements of β do not involve the subscripts r and s. Interchanging r and s, let P become P'. Now

$$P^2 = P'^2,$$

since each square is Δ, in which two rows and also two columns have been interchanged;

$$\therefore\ P = \pm P'.$$

But because of the interchange in r and s,

$$a_{rs}\beta \text{ becomes } -a_{rs}\beta;$$

or, since the term $a_{rs}\beta$ of $P = -a_{sr}\beta$ of P', it follows that $P = -P'$, as was to be shown.

129. We shall now prove a theorem by which we may compute Pfaffians of order $2n$ from those of order $2n-2$.*

Assuming

$$\sqrt{\beta_{ii}} = (-1)^i(2,\ 3,\ \cdots,\ i-1,\ i+1,\ \cdots,\ 2n), \tag{1}$$

or, after making $i-2$ cyclical interchanges,

$$\sqrt{\beta_{ii}} = (i+1,\ i+2,\ \cdots,\ 2n,\ 2,\ 3,\ \cdots,\ i-1), \tag{2}$$

where β has the same meaning as in **125**, we show that

$$\sqrt{\beta_{ii}}\sqrt{\beta_{kk}} = \beta_{ik}; \tag{3}$$

and then since

$$P = a_{12}\sqrt{\beta_{22}} + a_{13}\sqrt{\beta_{33}} + \cdots + a_{1\,2n}\sqrt{\beta_{2n\,2n}}, \tag{4}$$

---

* There is a difference in the nomenclature. We have here considered the order of the Pfaffian to be determined by the number of subscripts involved. Some authors determine the order of the Pfaffian by the order of the terms in the elements. Thus (1, 2, 3, 4), or $\|a_{14}|$, which we have designated as a Pfaffian of the *fourth* order, is said by some writers to be of the *second* order.

$$\left.\begin{array}{l}(1, 2, 3, \cdots, 2n) = a_{12}(3, \cdots, 2n) + a_{13}(4, \cdots, 2n, 2) + \cdots \\ \quad + a_{12n}(2, 3, \cdots, 2n-1)\end{array}\right\} \tag{5}$$

To show that upon the assumption (1) or (2) the equation (3) results, we proceed as follows:

Since $$\beta_{ii}\beta_{kk} = \beta_{ik}^2,$$

the terms of $\sqrt{\beta_{ii}}\sqrt{\beta_{kk}}$ must be equal each to each to the terms of β_{ik}, or equal with contrary sign. The product

$$\left.\begin{array}{r}(-1)^{i+k}(2, 3, \cdots, i-1, i+1, \cdots, 2n) \\ (2, 3, \cdots, k-1, k+1, \cdots, 2n)\end{array}\right\} \tag{6}$$

becomes, after a certain number of interchanges,

$$(k, p, q, r, \cdots, u, v)(p, q, r, s, \cdots, v, i), \tag{7}$$

where $p, q, r, \cdots, u, v$ denote the series of numbers

$$2, 3, \cdots, 2n,$$

exclusive of i, k. Again,

$$\beta_{ik} = (-1)^{i+k}\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2\,k-1} & a_{2\,k+1} & \cdots \\ a_{32} & a_{33} & \cdots & a_{3\,k-1} & a_{3\,k+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i-1\,2} & a_{i-1\,3} & \cdots & a_{i-1\,k-1} & a_{i-1\,k+1} & \cdots \\ a_{i+1\,2} & a_{i+1\,3} & \cdots & a_{i+1\,k-1} & a_{i+1\,k+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{8}$$

$$(a_{mm} = 0, \ a_{rs} = -a_{sr})$$

becomes, after the same number of interchanges as were employed to change (6) to (7),

$$\begin{vmatrix} a_{kp} & a_{kq} & a_{kr} & \cdots & a_{kv} & a_{ki} \\ a_{pp} & a_{pq} & a_{pr} & \cdots & a_{pv} & a_{pi} \\ a_{qp} & a_{qq} & a_{qr} & \cdots & a_{qv} & a_{qi} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{vp} & a_{vq} & a_{vr} & \cdots & a_{vv} & a_{vi} \end{vmatrix}. \tag{9}$$

Now the first term of the product of (7) is

$$a_{kp}\,a_{qr}\cdots a_{uv}\,a_{pq}\,a_{rs}\cdots a_{vi},$$

which is identical with the first term of the determinant (9). Whence the truth of (2) is established, and (5) gives the desired expansion of P. It is to be noted that the successive terms of P are written cyclically. For example, $\Delta^{(4)}$ being a skew symmetrical determinant of the fourth order,

$$\Delta^{(4)} \equiv P^2 = (1, 2, 3, 4)^2,$$

and $$(1, 2, 3, 4) = a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23}.$$

$$\Delta^{(6)} \equiv P^2 = (1, 2, 3, \cdots 6)^2,$$

$$\begin{aligned}(1, 2, \cdots 6) = a_{12}(3, 4, 5, 6) + a_{13}(4, 5, 6, 2) + a_{14}(5, 6, 2, 3)\\ + a_{15}(6, 2, 3, 4) + a_{16}(2, 3, 4, 5)\end{aligned}$$

$$\begin{aligned}&= a_{12}a_{34}a_{56} + a_{12}a_{35}a_{64} + a_{12}a_{36}a_{45}\\ &+ a_{13}a_{45}a_{62} + a_{13}a_{46}a_{25} + a_{13}a_{42}a_{56}\\ &+ a_{14}a_{56}a_{23} + a_{14}a_{52}a_{36} + a_{14}a_{53}a_{62}\\ &+ a_{15}a_{62}a_{34} + a_{15}a_{63}a_{42} + a_{15}a_{64}a_{23}\\ &+ a_{16}a_{23}a_{45} + a_{16}a_{24}a_{53} + a_{16}a_{25}a_{34}.\end{aligned}$$

130. The student must have already noticed the analogy between determinants and Pfaffians referred to above. The following notation, based upon this analogy, is interesting. Since the Pfaffian involves just half the elements of a skew symmetrical determinant like Δ of **124**, II., we write the Pfaffian

$$P \equiv \left|\begin{matrix} a_{12} & a_{13} & a_{14} & \cdots & a_{1\,2n-1} & a_{1\,2n} \\ & a_{23} & a_{24} & \cdots & a_{2\,2n-1} & a_{2\,2n} \\ & & a_{34} & \cdots & a_{3\,2n-1} & a_{3\,2n} \\ & & & \cdots & \cdots & \cdots \\ & & & & a_{2n-2\,2n-1} & a_{2n-2\,2n} \\ & & & & & a_{2n-1\,2n} \end{matrix}\right|,$$

which is shortened to

$$||a_{12}a_{23}a_{34}\cdots a_{2n-1\,2n}|, \text{ or to } ff(a_{1\,2n}), \text{ or to } ||a_{1\,2n}|.$$

In particular, we have for a Pfaffian of the third order

$$\left|\begin{matrix} a_1 & b_1 & c_1 \\ & b_2 & c_2 \\ & & c_3 \end{matrix}\right| \equiv ff(a_1 b_2 c_3) \equiv \|\, a_1 \quad b_2 \quad c_3 |.$$

We may accordingly write equation (p), at the end of **127**,

$$\sqrt{\Delta^{(4)}} = \|\, a_{14}|, \text{ or rather } \Delta^{(4)} = \|\, a_{14}|^2;$$

and the general equation would be

$$\Delta^{(2n)} = \|\, a_{1\,2n}|^2.$$

131. We must here conclude the discussion of Pfaffians with the theorem: *a bordered* skew symmetrical determinant is the product of two Pfaffians.*

From equation (b), **122**, II.,

$$\Delta^2_{a_{2n1}} = \Delta \cdot \Delta_{a_{11}, a_{2n2n}}.$$

$$\therefore \Delta_{a_{2n1}} = (1, 2, \cdots, 2n)(2, 3, \cdots, 2n-2,\ 2n-1), \qquad (1)$$

which proves the theorem when the determinant is of odd order.

Let $\Delta^{(n)}$ be a skew symmetrical determinant of odd order. $\Delta_{a_{ii}}$ is a skew symmetrical determinant of even order, and hence

$$\begin{aligned}\sqrt{\Delta_{a_{ii}}} &= (-1)^{i-1}(1, 2, \cdots, i-1, i+1, \cdots, n) \\ &= (i+1, \cdots, n, 1, 2, \cdots, i-1).\end{aligned}$$

Now $\Delta^{(n)}$ being zero, we have, by **60**,

$$\Delta^2_{a_{ik}} = \Delta_{a_{ii}} \Delta_{a_{kk}}.$$

$$\begin{aligned}\therefore \Delta_{a_{ik}} &= (i+1, \cdots, n, 1, 2, \cdots, i-1) \qquad (2) \\ &\quad\ (k+1, \cdots, n, 1, 2, \cdots, k-1),\end{aligned}$$

which proves that a bordered skew symmetrical determinant of even order is the product of two Pfaffians: for any minor $\Delta_{a_{ik}}$

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* A *bordered* skew symmetrical determinant is one in which the minor of one of the corner elements is skew symmetrical.

of a skew symmetrical determinant is evidently expressible as a bordered skew symmetrical determinant.

If

$$\Delta \equiv \left| a_{16} \right| \begin{matrix} a_{rs} = - a_{sr}, \\ a_{ii} = 0 \end{matrix} \quad \text{we find by (1),}$$

$$\Delta_{a_{61}} \equiv - \begin{vmatrix} a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ a_{32} & a_{33} & a_{34} & a_{35} & a_{36} \\ a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \end{vmatrix}_{a_{ii} = 0,\, a_{rs} = - a_{sr}}$$

$$= - (1, 2, 3, 4, 5, 6)\ (2, 3, 4, 5).$$

Again, if

$$\Delta \equiv \left| a_{15} \right| \begin{matrix} a_{rs} = - a_{sr}, \\ a_{ii} = 0 \end{matrix} \quad \text{we find by (2)}$$

$$\Delta_{a_{42}} \equiv \begin{vmatrix} a_{21} & a_{23} & a_{25} & a_{24} \\ a_{11} & a_{13} & a_{15} & a_{14} \\ a_{31} & a_{33} & a_{35} & a_{34} \\ a_{51} & a_{53} & a_{55} & a_{54} \end{vmatrix}_{a_{ii} = 0,\, a_{rs} = - a_{sr}} = (5, 1, 2, 3)\ (3, 4, 5, 1),$$

as the student can readily verify.

Circulants.

132. The resultant of

$$f(x) \equiv a_1 x^2 + a_2 x + a_3 = 0, \tag{1}$$

$$\phi(x) \equiv x^3 \qquad\qquad - 1 = 0, \tag{2}$$

by Sylvester's method (**92**) is

$$\begin{vmatrix} a_1 & a_2 & a_3 & 0 & 0 \\ 0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{vmatrix} \equiv R.$$

Now α_1, α_2, α_3 being the three roots of unity, it is evident (**94**) that

$$R = f(\alpha_1) f(\alpha_2) f(\alpha_3); \qquad (3)$$

or, denoting one of the imaginary cube roots of unity by α, the other is α^2, and we may write

$$\begin{aligned} R &= f(1) f(\alpha) f(\alpha^2) \\ &= (a_1 + a_2 + a_3)(a_1\alpha^2 + a_2\alpha + a_3)(a_1\alpha + a_2\alpha^2 + a_3), \end{aligned}$$

an equation exhibiting the factors of R.

133. R is evidently a symmetrical determinant formed from the elements a_1, a_2, a_3 in its first row, in such a way that the last element in every row is the first element in every succeeding row, and the other elements are written in order. Such a determinant is called a *Circulant.** The intimate connection of the Circulant of the third order with the cube roots of unity was shown in the last article. We shall now prove that, in general, the circulant of the nth order,

$$C \equiv C(a_1 a_2 \cdots a_n) \equiv \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-3} & a_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_3 & a_4 & a_5 & \cdots & a_1 & a_2 \\ a_2 & a_3 & a_4 & \cdots & a_n & a_1 \end{vmatrix}$$

is the product of all factors of the form

$$a_n\alpha_i^{n-1} + a_{n-1}\alpha_i^{n-2} + a_{n-2}\alpha_i^{n-3} + \cdots + a_3\alpha_i^2 + a_2\alpha_i + a_1 \equiv f(\alpha_i),$$

in which α_i is one of the nth roots of unity, and i accordingly takes successively all the values 1, 2, $\cdots$ n. In symbols, we are to show

$$\begin{aligned} C(a_1 a_2 a_3 \cdots a_n) &\equiv \prod_{i=1}^{i=n} (a_n\alpha_i^{n-1} + a_{n-1}\alpha_i^{n-2} + \cdots + a_2\alpha_i + a_1) \\ &\equiv f(\alpha_1) f(\alpha_2) f(\alpha_3) \cdots f(\alpha_n). \end{aligned}$$

Write another determinant of the nth order

* The Circulant is of frequent occurrence in the Theory of numbers.

$$\Delta \equiv \begin{vmatrix} 1 & a_1 & a_1^2 & a_1^3 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & a_2^3 & \cdots & a_2^{n-1} \\ 1 & a_3 & a_3^2 & a_3^3 & \cdots & a_3^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & a_n^3 & \cdots & a_n^{n-1} \end{vmatrix}.$$

Multiplying by rows,

$$C\Delta \equiv \begin{vmatrix} f(a_1) & f(a_2) & \cdots & f(a_{n-1}) & f(a_n) \\ a_1 f(a_1) & a_2 f(a_2) & \cdots & a_{n-1} f(a_{n-1}) & a_n f(a_n) \\ a_1^2 f(a_1) & a_2^2 f(a_2) & \cdots & a_{n-1}^2 f(a_{n-1}) & a_n^2 f(a_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1^{n-1} f(a_1) & a_2^{n-1} f(a_2) & \cdots & a_{n-1}^{n-1} f(a_{n-1}) & a_n^{n-1} f(a_n) \end{vmatrix}.$$

Factoring this product,

$$C\Delta = f(a_1) f(a_2) \cdots f(a_n) \Delta.$$

$$\therefore C = f(a_1) f(a_2) \cdots f(a_n)$$

$$= \prod_{i=1}^{i=n} (a_n a_i^{n-1} + a_{n-1} a_i^{n-2} + \cdots + a_2 a_i + a_1).$$

For an illustration, $\begin{vmatrix} x & 0 & 0 & 0 & y \\ y & x & 0 & 0 & 0 \\ 0 & y & x & 0 & 0 \\ 0 & 0 & y & x & 0 \\ 0 & 0 & 0 & y & x \end{vmatrix}$

$$= (x + \alpha_1 y)(x + \alpha_2 y)(x + \alpha_3 y)(x + \alpha_4 y)(x + \alpha_5 y)$$

$$= (x+y)\left(x + \left[\frac{\sqrt{5}-1}{4} + \frac{\sqrt{10+2\sqrt{5}}}{4}\sqrt{-1}\right]y\right)$$

$$\left(x + \left[\frac{\sqrt{5}-1}{4} - \frac{\sqrt{10+2\sqrt{5}}}{4}\sqrt{-1}\right]y\right)$$

$$\left(x + \left[-\frac{\sqrt{5}+1}{4} + \frac{\sqrt{10-2\sqrt{5}}}{4}\sqrt{-1}\right]y\right)$$

$$\left(x + \left[-\frac{\sqrt{5}+1}{4} - \frac{\sqrt{10-2\sqrt{5}}}{4}\sqrt{-1}\right]y\right)$$

$$= x^5 + y^5,$$

as was evident from the beginning.

134. The circulant of the fourth order

$$C \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_4 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_1 \end{vmatrix}$$

can be expressed as a circulant of the second order, as follows. We have

$$-C = \begin{vmatrix} a_1 & -a_2 & a_3 & -a_4 \\ a_3 & -a_4 & a_1 & -a_2 \\ a_4 & -a_1 & a_2 & -a_3 \\ a_2 & -a_3 & a_4 & -a_1 \end{vmatrix} = \begin{vmatrix} a_1 & a_4 & a_3 & a_2 \\ a_3 & a_2 & a_1 & a_4 \\ a_2 & a_1 & a_4 & a_3 \\ a_4 & a_3 & a_2 & a_1 \end{vmatrix}.$$

The first of these determinants is obtained by interchanging the second and third rows, and multiplying by $(-1)^2$; the second is obtained from the first by reversing the order of the rows, and then reversing the order of the columns.

Multiplying them together,

$$C^2 \equiv$$

$$\begin{vmatrix} a_1^2-2a_2a_4+a_3^2 & 2a_1a_3-a_2^2-a_4^2 & 0 & 0 \\ 2a_3a_1-a_4^2-a_2^2 & a_3^2-2a_4a_2+a_1^2 & 0 & 0 \\ 0 & 0 & 2a_4a_2-a_1^2-a_3^2 & a_2^2+a_4^2-2a_1a_3 \\ 0 & 0 & a_2^2-2a_1a_3+a_4^2 & 2a_2a_4-a_3^2-a_1^2 \end{vmatrix}.$$

Whence expressing C^2 as the product of two minors, and extracting the square root,

$$C = \begin{vmatrix} a_1a_1 - a_4a_2 + a_3a_3 - a_2a_4 & a_3a_1 - a_2a_2 + a_1a_3 - a_4a_4 \\ a_3a_1 - a_2a_2 + a_1a_3 - a_4a_4 & a_1a_1 - a_4a_2 + a_3a_3 - a_2a_4 \end{vmatrix},$$

as was to be shown.

The method employed in this special case is equally applicable to show that, in general, *a circulant of order* $2n$ *can be expressed as a circulant of the nth order.*

We add the following proof, however, which is based upon the fundamental property of circulants.

We have to show that

$$C \equiv \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{2n-1} & a_{2n} \\ a_{2n} & a_1 & a_2 & \cdots & a_{2n-2} & a_{2n-1} \\ a_{2n-1} & a_{2n} & a_1 & \cdots & \cdots & a_{2n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_3 & a_4 & a_5 & \cdots & a_1 & a_2 \\ a_2 & a_3 & a_4 & \cdots & a_{2n} & a_1 \end{vmatrix} = \begin{vmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \\ b_n & b_1 & \cdots & b_{n-2} & b_{n-1} \\ b_{n-1} & b_n & \cdots & b_{n-3} & b_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_3 & b_4 & \cdots & b_1 & b_2 \\ b_2 & b_3 & \cdots & b_n & b_1 \end{vmatrix}, \qquad (1)$$

where

$$\begin{aligned} b_1 &\equiv a_1 a_1 - a_{2n} a_2 + a_{2n-1} a_3 - + \cdots - a_2 a_{2n} \\ b_2 &\equiv a_3 a_1 - a_2 a_2 + a_1 a_3 - + \cdots - a_4 a_{2n} \\ b_3 &\equiv a_5 a_1 - a_4 a_2 + a_3 a_3 - + \cdots - a_6 a_{2n} \\ \cdots & \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ b_k &\equiv a_{2k-1} a_1 - a_{2k-2} a_2 + a_{2k-3} a_3 - + \cdots - a_{2k} a_{2n}. \end{aligned}$$

The first determinant

$$C = \prod_{i=1}^{i=2n} (a_{2n} \alpha_i^{2n-1} + a_{2n-1} \alpha_i^{2n-2} + a_{2n-2} \alpha_i^{2n-3} + \cdots + a_2 \alpha_i + a_1). \qquad (2)$$

Now for every $2n$th root α of unity there is one $-\alpha$. Hence (2) may be written

$$C = \prod_{i=1}^{i=n} (b_n \alpha_i^{2n-2} + b_{n-1} \alpha_i^{2n-4} + \cdots + b_3 \alpha_i^4 + b_2 \alpha_i^2 + b_1). \qquad (3)$$

If $\qquad \pm \alpha_1, \ \pm \alpha_2, \ \pm \alpha_3, \ \pm \alpha_4, \ \cdots, \ \pm \alpha_n,$

are the $2n$th roots of unity, it is evident that

$$\alpha_1^2, \ \alpha_2^2, \ \alpha_3^2, \ \cdots, \ \alpha_n^2,$$

are the nth roots of unity. Hence the second member of (3) equals the second determinant of (1), which establishes the theorem.

For example,

$$C \equiv \begin{vmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & a & a \end{vmatrix} = \begin{vmatrix} E & F \\ F & E \end{vmatrix},$$

in which

$$E \equiv a^2 + c^2 - 2\,bd, \quad F \equiv -b^2 - d^2 + 2\,ac.$$

$$\therefore\ C = (a^2 + c^2 - 2\,bd)^2 - (2\,ac - b^2 - d^2)^2.$$

Centro-symmetric Determinants.

135. If we suppose a determinant to be symmetrical with respect to the centre of the square (centro-symmetric*), we have, if the determinant is of order $2\,n$,

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1\,n-1} & a_{1n} & b_{11} & b_{12} & \cdots & b_{1\,n-1} & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2\,n-1} & a_{2n} & b_{21} & b_{21} & \cdots & b_{2\,n-1} & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{n\,n-1} & a_{nn} & b_{n1} & b_{n2} & \cdots & b_{n\,n-1} & b_{nn} \\ b_{nn} & b_{n\,n-1} & \cdots & b_{n\,2} & b_{n1} & a_{n\,n} & a_{n\,n-1} & \cdots & a_{n2} & a_{n1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{2n} & b_{2\,n-1} & \cdots & b_{2\,2} & b_{2\,1} & a_{2\,n} & a_{2\,n-1} & \cdots & a_{2\,2} & a_{2\,1} \\ b_{1n} & b_{1\,n-1} & \cdots & b_{1\,2} & b_{1\,1} & a_{1\,n} & a_{1\,n-1} & \cdots & a_{1\,2} & a_{1\,1} \end{vmatrix}.$$

We will transform Δ as follows: add the last column to the first, the $(2\,n-1)$th to the second, and so on, finally adding the $(n+1)$th to the nth. Afterward subtract the first row from the last, the second from the $(2\,n-1)$th, and so on, finally subtracting the nth from the $(n+1)$th. Then

$$\Delta = \begin{vmatrix} a_{11} + b_{1n} & \cdots & a_{1n} + b_{11} & b_{11} & \cdots & b_{1n} \\ a_{21} + b_{2n} & \cdots & a_{2n} + b_{21} & b_{21} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} + b_{nn} & \cdots & a_{nn} + b_{n1} & b_{n1} & \cdots & b_{nn} \\ 0 & \cdots & 0 & a_{nn} - b_{n1} & \cdots & a_{n1} - b_{nn} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_{2n} - b_{21} & \cdots & a_{21} - b_{2n} \\ 0 & \cdots & 0 & a_{1n} - b_{11} & \cdots & a_{11} - b_{1n} \end{vmatrix}.$$

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* It may be shown that the product of any two determinants of the nth order is expressible as a centro-symmetric determinant of the $2\,n$th order.

Hence

$$\Delta = \begin{vmatrix} a_{11}+b_{1n} & a_{12}+b_{1n-1} & \cdots & a_{1n-1}+b_{12} & a_{1n}+b_{11} \\ a_{21}+b_{2n} & a_{22}+b_{2n-1} & \cdots & a_{2n-1}+b_{22} & a_{2n}+b_{21} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}+b_{nn} & a_{n2}+b_{nn-1} & \cdots & a_{nn-1}+b_{n2} & a_{nn}+b_{n1} \end{vmatrix}$$

$$\times \begin{vmatrix} a_{nn}-b_{n1} & a_{nn-1}-b_{n2} & \cdots & a_{n2}-b_{nn-1} & a_{n1}-b_{nn} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n}-b_{21} & a_{2n-1}-b_{22} & \cdots & a_{22}-b_{2n-1} & a_{21}-b_{2n} \\ a_{1n}-b_{11} & a_{1n-1}-b_{12} & \cdots & a_{12}-b_{1n-1} & a_{11}-b_{1n} \end{vmatrix}.$$

If Δ is of order $2n+1$, we write

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} & k_1 & b_{11} & \cdots & b_{1n-1} & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} & k_2 & b_{21} & \cdots & b_{2n-1} & b_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & k_n & b_{n1} & \cdots & b_{nn-1} & b_{nn} \\ l_1 & l_2 & \cdots & l_n & r & l_n & \cdots & l_2 & l_1 \\ b_{nn} & b_{nn-1} & \cdots & b_{n1} & k_n & a_{nn} & \cdots & a_{n2} & a_{n1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{1n} & b_{1n-1} & \cdots & b_{11} & k_1 & a_{1n} & \cdots & a_{12} & a_{11} \end{vmatrix}.$$

By making just the same transformations as before, we find

$$\Delta = \begin{vmatrix} a_{11}+b_{1n} & a_{12}+b_{1n-1} & \cdots & a_{1n}+b_{11} & k_1 \\ a_{21}+b_{2n} & a_{22}+b_{2n-1} & \cdots & a_{2n}+b_{21} & k_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1}+b_{nn} & a_{n2}+b_{nn-1} & \cdots & a_{nn}+b_{n1} & k_n \\ 2l_1 & 2l_2 & \cdots & 2l_n & r \end{vmatrix}$$

$$\times \begin{vmatrix} a_{nn}-b_{n1} & a_{nn-1}-b_{n2} & \cdots & a_{n1}-b_{nn} \\ \cdots & \cdots & \cdots & \cdots \\ a_{2n}-b_{21} & a_{2n-1}-b_{22} & \cdots & a_{21}-b_{2n} \\ a_{1n}-b_{11} & a_{1n-1}-b_{12} & \cdots & a_{11}-b_{1n} \end{vmatrix}.$$

Collecting results, we have: *a centro-symmetric determinant equals the product of two determinants each of the nth order, if the order of the symmetric determinant is $2n$; if the order of the symmetric determinant is $2n+1$, the factors are of order n and $n+1$ respectively.*

For an illustration we expand the following determinant:

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ b & a & d & c & f & e & h & g \\ c & d & a & b & g & h & e & f \\ d & c & b & a & h & g & f & e \\ e & f & g & h & a & b & c & d \\ f & e & h & g & b & a & d & c \\ g & h & e & f & c & d & a & b \\ h & g & f & e & d & c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} a+h & b+g & c+f & d+e \\ b+g & a+h & d+e & c+f \\ c+f & d+e & a+h & b+g \\ d+e & c+f & b+g & a+h \end{vmatrix} \times \begin{vmatrix} a-h & b-g & c-f & d-e \\ b-g & a-h & d-e & c-f \\ c-f & d-e & a-h & b-g \\ d-e & c-f & b-g & a-h \end{vmatrix}$$

$$= \begin{vmatrix} a+h+d+e & b+g+c+f \\ b+g+c+f & a+h+d+e \end{vmatrix} \times \begin{vmatrix} a+h-d-e & b+g-c-f \\ b+g-c-f & a+h-d-e \end{vmatrix}$$

$$\times \begin{vmatrix} a-h+d-e & b-g+c-f \\ b-g+c-f & a-h+d-e \end{vmatrix} \times \begin{vmatrix} a-h-d+e & b-g-c+f \\ b-g-c+f & a-h-d+e \end{vmatrix}.$$

Continuants.

136. Consider the three simultaneous equations:

$$\left.\begin{array}{ll} (a) & 3x_1 - x_2 = 1 \\ (b) & x_1 + 4x_2 - x_3 = 0 \\ (c) & x_2 + 5x_3 = 0 \end{array}\right\}.$$

From (a),

$$x_1\left(3-\frac{x_2}{x_1}\right)=1; \qquad \therefore\ x_1=\cfrac{1}{3-\cfrac{x_2}{x_1}}.$$

From (b),

$$\frac{x_2}{x_1}=\cfrac{1}{\cfrac{x_3}{x_2}-4}; \qquad \therefore\ x_1=\cfrac{1}{3+\cfrac{1}{4-\cfrac{x_3}{x_2}}}.$$

From (c),

$$\frac{x_3}{x_2}=-\frac{1}{5}; \qquad \therefore\ x_1=\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5}}}.$$

The value of x_1 is thus expressed as a continued fraction. If we solve for x_1 by **69**, we find

$$x_1=\begin{vmatrix}1 & -1 & 0\\ 0 & 4 & -1\\ 0 & 1 & 5\end{vmatrix}\div\begin{vmatrix}3 & -1 & 0\\ 1 & 4 & -1\\ 0 & 1 & 5\end{vmatrix}.$$

We see then that a continued fraction may be expressed as the quotient of two determinants.

We shall now proceed to the application of determinants to continued fractions in general.

137. From the simultaneous equations

$$\text{I}\left\{\begin{array}{lllll}
(1) & a_1x_1-x_2 & =a_1 & & \\
(2) & a_2x_1+a_2x_2=x_3 & & & \\
(3) & & a_3x_2+a_3x_3=x_4 & & \\
 & \cdots & \cdots & \cdots & \cdots \\
(n-1) & & & a_{n-1}x_{n-2}+a_{n-1}x_{n-1}=x_n & \\
(n) & & & & a_nx_{n-1}\quad+a_nx_n=x_{n+1}
\end{array}\right.$$

we obtain from (1)

$$x_1 = \frac{a_1}{a_1 - \dfrac{x_2}{x_1}}.$$

Substituting in this the values of

$$\frac{x_2}{x_1}, \quad \frac{x_3}{x_2}, \quad \cdots \quad \frac{x_{n-1}}{x_{n-2}}, \quad \frac{x_n}{x_{n-1}},$$

as obtained from (2), (3), $\cdots$ $(n-1)$, (n), we have

$$x_1 = \cfrac{a_1}{a_1 + \cfrac{a_2}{a_2 + \cfrac{a_3}{a_3 + \ddots + \cfrac{a_{n-1}}{a_{n-1} + \cfrac{a_n}{a_n - \cfrac{x_{n+1}}{x_n}}}}}}.$$

The value of x_1 is seen to be expressible as a continued fraction. If we stop at the nth quotient, and thus take the nth convergent for the value of x_1, then x_{n+1} and all the succeeding x's must be conceived to vanish. In that case x_1 is the continued fraction.

$$F \equiv \frac{a_1}{a_1}\,\frac{a_2}{+a_2}\,\frac{a_3}{+a_3}\cdots\frac{a_{n-1}}{+a_n}\,\frac{a_n}{+a_n}.$$

The consecutive convergents to F will be denoted by

$$\frac{P_1}{Q_1}, \quad \frac{P_2}{Q_2}, \quad \cdots \quad \frac{P_n}{Q_n}.$$

The determinant expression for $\dfrac{P_n}{Q_n}$ is now found by making $x_{n+1} = 0$ in equations I., and solving for x_1 by **69**. We find

$$x_1 = \frac{\begin{vmatrix} a_1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & a_3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & a_4 & a_4 & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_{n-1} & a_{n-1} & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_n & a_n \end{vmatrix}}{\begin{vmatrix} a_1 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & a_3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_{n-1} & a_{n-1} & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a_n & a_n \end{vmatrix}},$$

which is the determinant expression sought, and hence is

$$\frac{P_n}{Q_n}.$$

Looking at numerator and denominator of this convergent, we see that

$$P_n = a_1 \frac{dQ_n}{da_1},$$

and thus

$$F = \frac{a_1 \dfrac{dQ_n}{da_1}}{Q_n}, \text{ and } \therefore F = a_1 \frac{d(\log Q_n)}{da_1}.$$

138. A determinant having the form of Q_n in the preceding article is called a *continuant*; *i.e.*, a continuant is a determinant in which the elements outside of the principal diagonal and the two adjacent minor diagonals are all zeros, and one of these minor diagonals has each of its elements -1.

Since

$$\begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & a_3 & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & a_{n-1} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n & a_n \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & a_2 & a_3 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & a_3 & a_4 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & a_{n-1} & a_n \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & a_n \end{vmatrix},$$

it is immaterial on which side of the principal diagonal we write that minor diagonal whose elements are -1. Also we may write

$$Q_n \equiv \begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_3 & a_3 & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & a_{n-1} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n & a_n \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a_2 & a_2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -a_3 & a_3 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -a_{n-1} & a_{n-1} & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -a_n & a_n \end{vmatrix}.$$

We shall employ the following notation:

$$Q_n \equiv \begin{pmatrix} a_2 a_3 \cdots a_n \\ a_1 a_2 a_3 \cdots a_n \end{pmatrix}.$$

Thus $$\begin{pmatrix} & a_2 & & a_3 & & a_4 & \\ a_1 & & a_2 & & a_3 & & a_4 \end{pmatrix} \equiv \begin{vmatrix} a_1 & -1 & 0 & 0 \\ a_2 & a_2 & -1 & 0 \\ 0 & a_3 & a_3 & -1 \\ 0 & 0 & a_4 & a_4 \end{vmatrix}.$$

Returning to $\dfrac{P_n}{Q_n}$, we may now write

$$\frac{P_n}{Q_n} \equiv \frac{\alpha_1 \begin{pmatrix} \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ a_2 & a_3 & \cdots & a_n \end{pmatrix}}{\begin{pmatrix} \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}};$$

or, *the nth convergent to a continued fraction F is expressible as the quotient of two continuants multiplied by the first numerator of F.*

139. Expanding P_n in terms of the elements of the last row, we find

$$P_n \equiv \alpha_1 \begin{pmatrix} \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ a_2 & a_3 & \cdots & a_n \end{pmatrix}$$

$$= a_n \alpha_1 \begin{vmatrix} a_2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_3 & a_3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_4 & a_4 & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_{n-2} & a_{n-2} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{n-1} & a_{n-1} \end{vmatrix}$$

$$+ \alpha_n \alpha_1 \begin{vmatrix} a_2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_3 & a_3 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha_4 & a_4 & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_{n-3} & a_{n-1} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha_{n-2} & a_{n-2} \end{vmatrix}$$

$$= a_n \alpha_1 \begin{pmatrix} \alpha_3 & \alpha_4 & \cdots & \alpha_{n-1} \\ a_2 & a_3 & \cdots & a_{n-1} \end{pmatrix} + \alpha_n \alpha_1 \begin{pmatrix} \alpha_3 & \alpha_4 & \cdots & \alpha_{n-2} \\ a_2 & a_3 & \cdots & a_{n-2} \end{pmatrix}$$

$$= a_n P_{n-1} + \alpha_n P_{n-2}. \qquad (A)$$

Similarly,

$$Q_n \equiv \begin{pmatrix} \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} = a_n \begin{pmatrix} \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} \\ a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}$$

$$+ \alpha_n \begin{pmatrix} \alpha_2 & \alpha_3 & \cdots & \alpha_{n-2} \\ a_1 & a_2 & \cdots & a_{n-2} \end{pmatrix}$$

$$= a_n Q_{n-1} + \alpha_n Q_{n-2}. \qquad (B)$$

140. It is to be observed that the equations (A) and (B), besides establishing the law of formation of the consecutive convergents to F, give the expansion of a continuant of order n in terms of continuants of lower orders. Thus, by (B),

$$\left(\begin{matrix} & \alpha_2 & \alpha_3 & \alpha_4 \\ a_1 & a_2 & a_3 & a_4 \end{matrix}\right) = a_4\left(\begin{matrix} & \alpha_2 & \alpha_3 \\ a_1 & a_2 & a_3 \end{matrix}\right) + \alpha_4\left(\begin{matrix} & \alpha_2 \\ a_1 & a_2 \end{matrix}\right),$$

$$= a_3a_4\left(\begin{matrix} & \alpha_2 \\ a_1 & a_2 \end{matrix}\right) + a_4\alpha_3\,(a_1) + \alpha_4\left(\begin{matrix} & \alpha_2 \\ a_1 & a_2 \end{matrix}\right),$$

$$= a_1a_2a_3a_4 + \alpha_2a_3a_4 + a_1\alpha_3a_4 + a_1a_2\alpha_4 + \alpha_2\alpha_4.$$

141. Equation (B) is, in fact, a special case of the more general theorem

$$\left(\begin{matrix} & \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ a_1 & a_2 & a_3 & & \cdots & a_n \end{matrix}\right) = \left(\begin{matrix} & \alpha_2 & \alpha_3 & \cdots & \alpha_r \\ a_1 & a_2 & & \cdots & a_r \end{matrix}\right)\left(\begin{matrix} & \alpha_{r+2} & \cdots & \alpha_n \\ a_{r+1} & & \cdots & a_n \end{matrix}\right)$$

$$+ \alpha_{r+1}\left(\begin{matrix} & \alpha_2 & \cdots & \alpha_{r-1} \\ a_1 & & \cdots & a_{r-1} \end{matrix}\right)\left(\begin{matrix} & \alpha_{r+3} & \cdots & \alpha_n \\ a_{r+2} & & \cdots & a_n \end{matrix}\right).$$

This is easily proved by writing out the continuant of the first member in full, and expanding by Laplace's Theorem (**55**) in terms of the minors formed from the first r rows and their complementaries.

We may also use (B) to obtain another expansion of Q_n. Thus

$$\left(\begin{matrix} & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ a_1 & a_2 & & \cdots & a_n \end{matrix}\right) = a_1\left(\begin{matrix} & \alpha_3 & \alpha_4 & \cdots & \alpha_n \\ a_2 & a_3 & & \cdots & a_n \end{matrix}\right) + \alpha_2\left(\begin{matrix} & \alpha_4 & \alpha_5 & \cdots & \alpha_n \\ a_3 & \cdots & & \cdots & a_n \end{matrix}\right), \tag{C}$$

as the student may easily verify by expanding the first member in terms of the elements of the first column.

142. It will afford the student an excellent exercise to take the quotient

$$\frac{a_1\left(\begin{matrix} & \alpha_3 & \cdots & \alpha_n \\ a_2 & a_3 & \cdots & a_n \end{matrix}\right)}{\left(\begin{matrix} & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ a_1 & a_2 & & \cdots & a_n \end{matrix}\right)},$$

and, with the help of (C) of the preceding article, transform it into the continued fraction

$$\frac{\alpha_1}{a_1}\;\frac{\alpha_2}{+a_2}\;\frac{\alpha_3}{+a_3}\cdots\frac{\alpha_n}{+a_n}.$$

143. **57** or **62** established the theorem

$$\Delta\frac{d^2\Delta}{da_{ik}\,da_{pe}}=\frac{d\Delta}{da_{ik}}\,\frac{d\Delta}{da_{pe}}-\frac{d\Delta}{da_{pk}}\,\frac{d\Delta}{da_{ie}}. \tag{1}$$

Let $$\Delta\equiv Q_n\equiv\left(\begin{matrix} & \alpha_2 & \cdots & \alpha_n & \\ a_1 & a_2 & \cdots & & a_n\end{matrix}\right),$$

and let $$i=k=1,\quad p=e=n.$$

Then $$\frac{d^2\Delta}{da_{11}\,da_{nn}}=\left(\begin{matrix} & \alpha_3 & \cdots & \alpha_{n-1} & \\ a_2 & a_3 & \cdots & & a_{n-1}\end{matrix}\right)\equiv\frac{P_{n-1}}{\alpha_1},$$

where P_{n-1} has the meaning assigned to it from the beginning.

Also $$\frac{d\Delta}{da_{11}}=\left(\begin{matrix} & \alpha_3 & \cdots & \alpha_n & \\ a_2 & a_3 & \cdots & & a_n\end{matrix}\right)\equiv\frac{P_n}{\alpha_1}.$$

Similarly,

$$\frac{d\Delta}{da_{1n}}=\alpha_2\alpha_3\cdots\alpha_n;\quad \frac{d\Delta}{da_{n1}}=(-1)^{n-1};$$

and $$\frac{d\Delta}{da_{nn}}=\left(\begin{matrix} & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \\ a_1 & a_2 & & \cdots & & a_{n-1}\end{matrix}\right)\equiv Q_{n-1};$$

$$\therefore\left(\begin{matrix} & \alpha_2 & \alpha_3 & \cdots & \alpha_n & \\ a_1 & a_2 & & \cdots & & a_n\end{matrix}\right)\left(\begin{matrix} & \alpha_3 & \alpha_4 & \cdots & \alpha_{n-1} & \\ a_2 & a_3 & & \cdots & & a_{n-1}\end{matrix}\right)$$

$$=\left(\begin{matrix} & \alpha_3 & \cdots & \alpha_n & \\ a_2 & a_3 & \cdots & & a_n\end{matrix}\right)\left(\begin{matrix} & \alpha_2 & \cdots & \alpha_{n-1} & \\ a_1 & a_2 & \cdots & & a_{n-1}\end{matrix}\right)-(-1)^{n-1}\alpha_2\alpha_3\cdots\alpha_n;$$

or $$Q_nP_{n-1}-P_nQ_{n-1}=(-1)^n\alpha_1\alpha_2\alpha_3\cdots\alpha_n.$$

144. With the help of determinants we may now show that

$$F \equiv \frac{a_1}{a_1}\,\frac{a_2}{+a_2}\,\frac{a_3}{+a_3}\cdots\frac{a_{k-1}}{+a_{k-1}}\,\frac{a_k}{+a_k}\,\frac{a_{k+1}}{+a_{k+1}}\cdots\frac{a_n}{+a_n}$$

equals

$$F' \equiv \frac{a_1}{a_1}\,\frac{a_2}{+a_2}\,\frac{a_3}{+a_3}\cdots\frac{xa_{k-1}}{+xa_{k-1}}\,\frac{xa_k}{+a_k}\,\frac{a_{k+1}}{+a_{k+1}}\cdots\frac{a_n}{+a_n}.$$

Express F as the quotient of two determinants, employing the form obtained in **137**. Now transform numerator and denominator of this determinant as follows. Multiply the $(k-1)$th column, and divide the $(k-1)$th row of each determinant by $-a_{k-1}$. Then perform the same operations upon each succeeding row and column. Afterward multiply the $(k-1)$th row of each determinant by x. The value of the fraction is not changed, and we obtain for the value of F

$$\frac{\begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_3 & a_3 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{k-2} & a_{k-1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -x & a_{k-1}x & -x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_k & a_k & -1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & a_n & a_n \end{vmatrix}}{\begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & a_2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_3 & a_3 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{k-2} & a_{k-1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -x & a_{k-1}x & -x & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_k & a_k & -1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & a_n & a_n \end{vmatrix}}.$$

Now divide the $(k-1)$th column, and multiply the $(k-1)$th row of numerator and denominator by $-a_{k-1}$. Then divide the kth column, and multiply the kth row of numerator and denominator by $-a_{k-1}x$, and perform the same operations upon the succeeding rows and columns. We obtain

$$\frac{\begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_3 & a_3 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{k-1}x & a_{k-1}x & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_k x & a_k & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{k+1} & a_{k+1} & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & a_n & a_n \end{vmatrix}}{\begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & a_2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_3 & a_3 & -1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{k-1}x & a_{k-1}x & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_k x & a_k & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{k+1} & a_{k+1} & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & a_n & a_n \end{vmatrix}}$$

But this quotient is the continued fraction F'.

145. In a certain investigation it becomes necessary to show that the denominators D_1 and D_2 of the convergents to the fractions

$$\frac{b}{a_1}\,\frac{b}{+a_2}\cdots\frac{b}{+a_n} \quad\text{and}\quad \frac{b}{a_n}\,\frac{b}{+a_{n-1}}\cdots\frac{b}{+a_1}$$

are equal.

We have

$$D_1 \equiv \begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b & a_2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b & a_3 & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b & a_{n-1} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & a_n \end{vmatrix},$$

and

$$D_2 \equiv \begin{vmatrix} a_n & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b & a_{n-1} & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & b & a_{n-2} & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & b & a_2 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & b & a_1 \end{vmatrix}.$$

By reversing the order of the columns in D_2, and also the order of the rows, and afterward making the rows the columns in order, the original determinant is unchanged either in sign or magnitude. But by these transformations D_2 is changed to D_1. Whence $D_2 = D_1$, as was to be shown.

146. The quotient

$$\frac{|b_2 \ c_3|}{|a_1 \ b_2 \ c_3|}$$

can be expressed as a continued fraction, as follows:

$$\frac{|b_2 \ c_3|}{|a_1 \ b_2 \ c_3|} = \frac{\begin{vmatrix} 1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_2 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 0 & 0 \\ a_2 & b_2 & |b_1 c_2| \\ a_3 & b_3 & |b_1 c_3| \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & |b_1 c_2| \\ a_3 & b_3 & |b_1 c_3| \end{vmatrix}},$$

$$= \frac{\begin{vmatrix} b_3 & 0 & 0 \\ |a_2b_3| & b_2 & |b_1c_2| \\ 0 & b_3 & |b_1c_3| \end{vmatrix}}{\begin{vmatrix} |a_1b_3| & b_1 & 0 \\ |a_2b_3| & b_2 & |b_1c_2| \\ 0 & b_3 & |b_1c_3| \end{vmatrix}} = \frac{\begin{vmatrix} b_3 & 0 & 0 \\ |a_2b_3| & b_2 & -1 \\ 0 & -b_3|b_1c_2| & |b_1c_3| \end{vmatrix}}{\begin{vmatrix} |a_1b_3| & -1 & 0 \\ -b_1|a_2b_3| & b_2 & -1 \\ 0 & -|b_1c_2| & |b_1c_3| \end{vmatrix}}$$

$$= \cfrac{b_3}{|a_1b_3| - \cfrac{b_1|a_2b_3|}{b_2 - \cfrac{b_3|b_1c_2|}{|b_1c_3|}}}.$$

This process is equally applicable to show that, in general, the quotient of two determinants $\frac{\Delta_1}{\Delta_2}$ is expressible as a continued fraction, provided only that

$$\Delta_1 = \frac{d\Delta_2}{da_{11}}, \quad \text{or} \quad \Delta_1 = \frac{d\Delta_2}{da_{1n}}, \quad \text{or} \quad \Delta_1 = \frac{d\Delta_2}{da_{n1}}, \quad \text{or} \quad \Delta_1 = \frac{d\Delta_2}{da_{nn}}.$$

147. The continued fraction

$$F_1 \equiv m + \frac{\alpha_1}{a_1} \; \frac{\alpha_2}{+a_2} \; \frac{\alpha_3}{+a_3} \; \cdots \; \frac{\alpha_n}{+a_n}$$

is evidently equal to
$$\frac{\begin{pmatrix} & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ m & a_1 & a_2 & \cdots & a_n \end{pmatrix}}{\begin{pmatrix} \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}}$$

For

$$F_1 = \frac{\begin{vmatrix} \alpha_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_3 & a_3 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_n & a_n \end{vmatrix} + \begin{vmatrix} m & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_2 & a_2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & \alpha_3 & a_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha_n & a_n \end{vmatrix}}{\begin{pmatrix} \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix}}.$$

But the first determinant in the numerator may be written

$$\begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & a_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_n & a_n \end{vmatrix} = \begin{vmatrix} 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a_1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & a_2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & a_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_n & a_n \end{vmatrix};$$

whence the desired result is at once obtained by substituting in the numerator of the value of F_1, and adding the determinants.

148. We may, with the help of the preceding article, express the value of the periodic continued fraction

$$m + \frac{b_1}{\underset{*}{a_1}}\;\frac{b_2}{+a_2}\;\frac{b_3}{+a_3}\;\cdots\;\frac{b_3}{+a_2}\;\frac{b_2}{+a_1}\;\frac{b_1}{\underset{*}{+2m}}\;\cdots$$

as the quotient of two determinants. (The * marks the recurring period.)

If we put x for the continued fraction, we have

$$x = m + \frac{b_1}{a_1}\;\frac{b_2}{+a_2}\;\frac{b_3}{+a_3}\;\cdots\;\frac{b_3}{+a_2}\;\frac{b_2}{+a_1}\;\frac{b_1}{m+x}.$$

Then, by **147**,

$$x = \frac{\left(\begin{matrix} & b_1 & & b_2 & & \cdots & & b_3 & & b_2 & & b_1 & \\ m & & a_1 & & a_2 & \cdots & a_2 & & a_1 & & & & (m+x) \end{matrix}\right)}{\left(\begin{matrix} & b_2 & & b_3 & & \cdots & & b_2 & & b_1 & \\ a_1 & & a_2 & & a_3 & \cdots & a_2 & & a_1 & & (m+x) \end{matrix}\right)};$$

clearing of fractions, and expanding,

$$x\left(\begin{matrix} & b_2 & & b_3 & \cdots & & b_3 & & b_2 & & b_1 & \\ a_1 & & a_2 & & \cdots & a_2 & & a_1 & & m & \end{matrix}\right) + x^2\left(\begin{matrix} & b_2 & & b_3 & \cdots & & b_3 & & b_2 & \\ a_1 & & a_2 & & \cdots & & & a_2 & & a_1 \end{matrix}\right)$$

$$= \left(\begin{matrix} & b_1 & & b_2 & & \cdots & & b_2 & & b_1 & \\ m & & a_1 & & a_2 & \cdots & a_2 & & a_1 & & m \end{matrix}\right) + x\left(\begin{matrix} & b_1 & & b_2 & & \cdots & & b_2 & \\ m & & a_1 & & a_2 & \cdots & a_2 & & a_1 \end{matrix}\right).$$

But the first term of the first member equals the second term of the second member of this equation.

$$\therefore\ x = \pm\left[\frac{\begin{pmatrix} & b_1 & b_2 & \cdots & b_3 & b_2 & b_1 & \\ m & a_1 & a_2 & \cdots & a_3 & a_2 & a_1 & m\end{pmatrix}}{\begin{pmatrix} & b_2 & b_3 & \cdots & b_3 & b_2 & \\ a_1 & a_2 & a_3 & \cdots & a_3 & a_2 & a_1\end{pmatrix}}\right]^{\frac{1}{2}}.$$

149. Let us now consider the ascending continued fraction

$$F' \equiv \cfrac{\alpha_1 + \cfrac{\alpha_2 + \cfrac{\alpha_3 + \cdot\cdot\cdot + \cfrac{\alpha_n}{a_n}}{a_3}}{a_2}}{a_1} \equiv \frac{\alpha_1}{a_1}\ \frac{+\alpha_2}{a_2}\ \frac{+\alpha_3}{a_3} \cdots \frac{+\alpha_n}{a_n}.$$

Denote the convergents to F' by

$$\frac{p}{q},\quad \frac{p_1}{q_1},\quad \frac{p_2}{q_2},\quad \cdots\quad \frac{p_n}{q_n},$$

and let us obtain the determinant expression for the nth convergent.

We have evidently

$$q_n = a_1 a_2 a_3 \cdots a_n.$$

p_n is determined from the following equations, which the student can easily deduce:

$$\begin{array}{lr} p_1 & = \alpha_1 \\ -a_2p_1 + p_2 & = \alpha_2 \\ \quad -a_3p_2 + p_3 & = \alpha_3 \\ \cdots \qquad \cdots \qquad \cdots & \cdots \\ \qquad -a_{n-2}p_{n-3} + p_{n-2} & = \alpha_{n-2} \\ \qquad\quad -a_{n-1}p_{n-2} + p_{n-1} & = \alpha_{n-1} \\ \qquad\qquad a_n p_{n-1} + p_n & = \alpha_n. \end{array}$$

From these equations

$$p_n = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & a_1 \\ -a_2 & 1 & 0 & \cdots & 0 & 0 & 0 & a_2 \\ 0 & -a_3 & 1 & \cdots & 0 & 0 & 0 & a_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -a_{n-2} & 1 & 0 & a_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-1} & 1 & a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & a_n & a_n \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_2 & a_2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ a_3 & 0 & a_3 & -1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-2} & 0 & 0 & 0 & \cdots & a_{n-2} & -1 & 0 \\ a_{n-1} & 0 & 0 & 0 & \cdots & 0 & a_{n-1} & -1 \\ a_n & 0 & 0 & 0 & \cdots & 0 & 0 & a_n \end{vmatrix}.$$

$$\therefore \frac{p_n}{q_n} = \frac{\begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ a_2 & a_2 & -1 & 0 & \cdots & 0 & 0 \\ a_3 & 0 & a_3 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & 0 & 0 & 0 & \cdots & a_{n-1} & -1 \\ a_n & 0 & 0 & 0 & \cdots & 0 & a_n \end{vmatrix}}{\begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & -1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a_n \end{vmatrix}}.$$

150. The numerator and denominator of $\frac{p_n}{q_n}$ can be transformed into continuants, and thus the fraction F' can be transformed into a descending continued fraction, as follows:

Multiply the last row of p_n by a_{n-1}, and subtract from it the $(n-1)$th row multiplied by a_n; then multiply the $(n-1)$th row by a_{n-2}, and subtract from it the $(n-2)$th row multiplied by a_{n-1}; and so on. Then

$$p_n = \frac{\begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_2a_1+a_2 & -a_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -a_2a_3 & a_3a_2+a_3 & -a_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -a_{n-2}a_{n-1} & a_{n-1}a_{n-2}+a_{n-1} & -a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -a_{n-1}a_n & a_na_{n-1}+a_n \end{vmatrix}}{a_{n-1}a_{n-2}a_{n-3}\cdots a_3a_2a_1}.$$

Similarly,

$$q_n = \frac{\begin{vmatrix} a_1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -a_1a_2 & a_2a_1+a_2 & -a_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -a_2a_3 & a_3a_2+a_3 & -a_2 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -a_{n-2}a_{n-1} & a_{n-1}a_{n-2}+a_{n-1} & -a_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -a_{n-1}a_n & a_na_{n-1}+a_n \end{vmatrix}}{a_{n-1}a_{n-2}a_{n-3}\cdots a_2a_1}.$$

Whence, by **144**,

$$\frac{p_n}{q_n}$$

$$= \frac{a_1}{a_1}\;\frac{a_1a_2}{-a_2a_1+a_2}\;\frac{a_2a_1a_3}{-a_3a_2+a_3}\cdots\frac{a_{n-2}a_{n-3}a_{n-1}}{-a_{n-1}a_{n-2}+a_{n-1}}\;\frac{a_{n-1}a_{n-2}a_n}{-a_na_{n-1}+a_n},$$

the descending fraction sought.

Alternants.

151. Consider the determinant

$$\Delta \equiv \begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{vmatrix},$$

and the product

$$\begin{aligned} P \equiv (a_2-a_1)(a_3-a_1)(a_4-a_1) & \cdots (a_n-a_1) \\ \times (a_3-a_2)(a_4-a_2) & \cdots (a_n-a_2) \\ \times (a_4-a_3) & \cdots (a_n-a_3) \\ \cdots \quad & \cdots \quad \cdots \\ & \times (a_n-a_{n-1}) \end{aligned}$$

of the $\frac{n}{2}(n-1)$ differences of the n different quantities involved in Δ. This product is called the *difference product* of the n quantities $a_1, a_2, \cdots a_n$, and for it the notation $\zeta^{\frac{1}{2}}(a_1, a_2, a_3, \cdots a_n)$ has been adopted.

The reader will remember that the square of the difference product was denoted by $\zeta(a_1, a_2, \cdots a_n)$, and thus the difference product itself is very appropriately designated by $\zeta^{\frac{1}{2}}(a_1, a_2, \cdots a_n)$.

We shall now show that

$$\Delta \equiv P \equiv \zeta^{\frac{1}{2}}(a_1, a_2, \cdots, a_n). \tag{1}$$

If in Δ we put $a_i = a_k$, Δ vanishes; hence Δ is divisible by each factor of P, and hence by P. Again, Δ and P are each polynomials of degree $\frac{n}{2}(n-1)$, and therefore

$$\Delta = \lambda\zeta^{\frac{1}{2}}(a_1, a_2, a_3, \cdots, a_n),$$

where λ is a factor independent of $a_1, a_2, \cdots a_n$. From the special case

$$\Delta \equiv \begin{vmatrix} 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \\ 1 & a_3 & a_3^2 \end{vmatrix} = (a_2-a_1)(a_3-a_1)(a_3-a_2),$$

we see that $\lambda = 1$, and thus the truth of (1) is established.*

152. Δ of the preceding article is evidently an alternating function; for the interchange of a_i and a_k amounts to an interchange of two rows in the determinant, and hence changes its sign. Δ is accordingly an *Alternant*. In general, an alternant is a determinant in which each element of the first row is a function of x_1, the corresponding elements of the second row the same functions of x_2, and so on. Thus

$$\Delta \equiv \begin{vmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{vmatrix} \equiv A[f_1(x_1),\ f_2(x_2)\ \cdots\ f_n(x_n)]$$

is an alternant.

153. We can easily show that

$$\Delta \equiv \begin{vmatrix} 1 & f_1(x_1) & f_2(x_1) & \cdots & f_{n-1}(x_1) \\ 1 & f_1(x_2) & f_2(x_2) & \cdots & f_{n-1}(x_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & f_1(x_n) & f_2(x_n) & \cdots & f_{n-1}(x_n) \end{vmatrix} = \lambda\zeta^{\frac{1}{2}}(x_1,\ x_2,\ \cdots\ x_n),$$

where $f_r(x)$ is a function of the rth degree in x, and λ is the product of the coefficients of the terms of highest degree in the several functions. For subtracting the first column multiplied by the proper number from the second, we reduce the elements of the second column to $p_1x_1,\ p_1x_2,\ p_1x_3,\ \cdots\ p_1x_n$. Then subtracting the sum of the first and second columns, each multiplied by the proper number, from the third column, the elements of this column become $p_2x_1^2,\ p_2x_2^2,\ \cdots\ p_2x_n^2$. Proceeding in this way, we see that finally

$$\Delta = \lambda\zeta^{\frac{1}{2}}(x_1,\ x_2,\ \cdots\ x_n),$$

---

* See also examples 6 and 7, page 37.

where $$\lambda = p_1 \cdot p_2 \cdots p_n.$$

For an example, putting

$$f_r(x) \equiv \frac{x(x-1)(x-2)\cdots(x-r+1)}{r\,!},$$

we have

$$\Delta \equiv \begin{vmatrix} 1 & f_1(x_1) & f_2(x_1) & \cdots & f_{n-1}(x_1) \\ 1 & f_1(x_2) & f_2(x_2) & \cdots & f_{n-1}(x_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & f_1(x_n) & f_2(x_n) & \cdots & f_{n-1}(x_n) \end{vmatrix} = \frac{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)}{(n-1)!\,(n-2)!\cdots 2\,!}.$$

154. Every alternant whose elements are rational integral functions of $x_1, x_2, \cdots x_n$, is divisible by $\zeta^{\frac{1}{2}}(x_1, x_2, x_3, \cdots x_n)$, and the quotient is a symmetric function of the variables. For the alternant vanishes if $x_i = x_k$, and hence is divisible by $x_i - x_k$, and thus by $\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)$. The quotient must be a symmetric function, for the interchange of x_i and x_k changes the sign of both dividend and divisor; therefore the sign of the quotient remains unchanged upon the interchange of two of the variables, and is accordingly a symmetric function. We shall now actually perform the division just considered. Alternants whose functions are powers of the variables are called *simple alternants*, and are of frequent occurrence. We proceed first to the discussion of simple alternants.

155. The quotient

$$\begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^q \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^q \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & \cdots & x_n^{n-2} & x_n^q \end{vmatrix} \div \zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n) \equiv \frac{A(x_1^0, x_2, x_3^2 \cdots x_{n-1}^{n-2}, x_n^q)}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)}$$

may be developed as follows:

Expand the dividend Δ in terms of the elements of the last column, and we obtain

$$A(x_1^0, x_2^1, x_3^2, \cdots x_{n-1}^{n-2}, x_n^q)$$

$$= x_1^q \frac{d\Delta}{dx_1^q} + x_2^q \frac{d\Delta}{dx_2^q} + \cdots + x_r^q \frac{d\Delta}{dx_r^q} + \cdots + x_n^q \frac{d\Delta}{dx_n^q}. \qquad (1)$$

Now, it is evident that each of the minors in this expansion is a difference product.

Thus

$$\frac{d\Delta}{dx_r^q} \equiv (-1)^{n+r} \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{r-1} & x_{r-1}^2 & \cdots & x_{r-1}^{n-2} \\ 1 & x_{r+1} & x_{r+1}^2 & \cdots & x_{r+1}^{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} \end{vmatrix}$$

$$\equiv (-1)^{n+r} \zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_{r-1}, x_{r+1}, \cdots x_n). \qquad (2)$$

Substituting in (1) the values of the minors as found from (2), and dividing both members of (1) by $\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)$, we have a series of terms, of which

$$\frac{(-1)^{n+r} x_r^q}{(x_n - x_r)(x_{n-1} - x_r) \cdots (x_{r+1} - x_r)(x_r - x_{r-1}) \cdots (x_r - x_2)(x_r - x_1)}$$

is the type. Thus we find

$$\frac{A(x_1^0, x_2^1, x_3^2, x_{n-1}^{n-2}, x_n^q)}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_n)}$$

$$= \overset{r=n}{\underset{r=1}{S}} \frac{(-1)^{n+r} x_r^q}{(x_n - x_r)(x_{n-1} - x_r) \cdots (x_{r+1} - x_r)(x_r - x_{r-1}) \cdots (x_r - x_1)};$$

or
$$= \frac{x_1^q}{(x_1 - x_n)(x_1 - x_{n-1}) \cdots (x_1 - x_2)}$$

$$+ \frac{x_2^q}{(x_2 - x_n)(x_2 - x_{n-1}) \cdots (x_2 - x_3)(x_2 - x_1)}$$

$$+ \cdots + \frac{x_{n-2}^q}{(x_{n-2} - x_n)(x_{n-2} - x_{n-1})(x_{n-2} - x_{n-3}) \cdots (x_{n-2} - x_1)}$$

$$+\frac{x_{n-1}^q}{(x_{n-1}-x_n)(x_{n-1}-x_{n-2})(x_{n-1}-x_{n-3})\cdots(x_{n-1}-x_1)}$$

$$+\frac{x_n^{\,q}}{(x_n-x_{n-1})(x_n-x_{n-2})\cdots(x_n-x_2)(x_n-x_1)}.$$

For an illustration, we have

$$\begin{vmatrix} 1 & 2 & 16 \\ 1 & 3 & 81 \\ 1 & 5 & 625 \end{vmatrix} \div \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \end{vmatrix}$$

$$=\frac{16}{(2-5)(2-3)}+\frac{81}{(3-5)(3-2)}+\frac{625}{(5-3)(5-2)}=69.$$

156. With the help of the preceding article we may reduce the quotient

$$\frac{A(x_1^0,\, x_2^1,\, x_3^2,\, \cdots\, x_{n-1}^{n-2},\, x_n^{\,q})}{\zeta^{\frac{1}{2}}(x_1,\, x_2,\, \cdots\, x_n)}$$

to the sum of two similar and simpler quotients, as follows:

Since

$$A(x_1^0,\, x_2^1,\, x_3^2,\, \cdots\, x_{n-1}^{n-2},\, x_n^{\,q}) - x_nA(x_1^0,\, x_2^1,\, x_3^2,\, \cdots\, x_{n-1}^{n-2},\, x_n^{\,q-1})$$

$$=\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{\,n-2} & x_1^{\,q-1}(x_1-x_n) \\ 1 & x_2 & x_2^2 & \cdots & x_2^{\,n-2} & x_2^{\,q-1}(x_2-x_n) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-2} & x_{n-1}^{q-1}(x_{n-1}-x_n) \\ 1 & x_n & x_n^{\,2} & \cdots & x_n^{n-2} & 0 \end{vmatrix}, \qquad (1)$$

we have, after dividing both members of (1) by $\zeta^{\frac{1}{2}}(x_1,\, x_2,\, \cdots\, x_n)$ in accordance with **155**, and striking out the factor common to numerator and denominator of each term in the second member,

$$\frac{A(x_1^0,\, x_2^1,\, x_3^2,\, \cdots,\, x_{n-1}^{n-2},\, x_n^{\,q})}{\zeta^{\frac{1}{2}}(x_1,\, x_2,\, \cdots,\, x_n)}-\frac{x_nA(x_1^0,\, x_2^1,\, x_3^2,\, \cdots\, x_{n-1}^{n-2},\, x_n^{\,q-1})}{\zeta^{\frac{1}{2}}(x_1,\, x_2,\, \cdots\, x_n)}$$

$$=\frac{x_1^{q-1}}{(x_1-x_2)(x_1-x_3)\cdots(x_1-x_{n-1})}+\frac{x_2^{q-1}}{(x_2-x_{n-1})\cdots(x_2-x_3)(x_2-x_1)}$$

$$+\cdots+\frac{x_{n-1}^{q-1}}{(x_{n-1}-x_{n-2})(x_{n-1}-x_{n-3})\cdots(x_{n-1}-x_1)}.$$

But the sum in the second member of this equation is, by **155**,

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-3} & x_1^{q-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-3} & x_2^{q-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & x_{n-1} & x_{n-1}^2 & \cdots & x_{n-1}^{n-3} & x_{n-1}^{q-1} \end{vmatrix} \div \zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_{n-1}).$$

Transposing, we have

$$\frac{A(x_1^0, x_2^1, x_3^2, \cdots x_{n-1}^{n-2}, x_n^q)}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)}=\frac{x_nA(x_1^0, x_2^1, x_3^2, \cdots x_{n-1}^{n-2}, x_n^{q-1})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)}$$

$$+\frac{A(x_1^0, x_2^1, x_3^2, \cdots x_{n-2}^{n-3}, x_{n-1}^{q-1})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_{n-1})},$$

which is the desired reduction. For example,

$$\frac{\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}}{\zeta^{\frac{1}{2}}(x, y, z)}=z\frac{\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}}{\zeta^{\frac{1}{2}}(x, y, z)}+\frac{\begin{vmatrix} 1 & x^2 \\ 1 & y_2 \end{vmatrix}}{\zeta^{\frac{1}{2}}(x, y)}=x+y+z\equiv\Sigma x.$$

$$\frac{\begin{vmatrix} 1 & x & x^4 \\ 1 & y & y^4 \\ 1 & z & z^4 \end{vmatrix}}{\zeta^{\frac{1}{2}}(x, y, z)}=z\frac{\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}}{\zeta^{\frac{1}{2}}(x, y, z)}+\frac{\begin{vmatrix} 1 & x^3 \\ 1 & y^3 \end{vmatrix}}{\zeta^{\frac{1}{2}}(x, y)}=z\Sigma x+y^2+xy+x^2$$
$$\equiv\Sigma x^2+\Sigma xy.$$

The student may show

$$\frac{\begin{vmatrix} 1 & x & x^5 \\ 1 & y & y^5 \\ 1 & z & z^5 \end{vmatrix}}{\zeta^{\frac{1}{2}}(x, y, z)}=\Sigma x^3+\Sigma x^2y+xyz.$$

$$\frac{\begin{vmatrix} 1 & x & x^6 \\ 1 & y & y^6 \\ 1 & z & z^6 \end{vmatrix}}{\zeta^{\frac{1}{2}}(x, y, z)} = \Sigma x^4 + \Sigma x^3 y + \Sigma x^2 y^2 + \Sigma x^2 yz.$$

157. Since every term of $\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)$ contains a permutation of all the powers of the variables from 1 to $n-1$, each term is of the $\frac{n}{2}(n-1)$th degree. Similarly, every term of $A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-1}^{n-2}, x_n^q)$ is of degree $\frac{(n-1)(n-2)}{2} + q$. Hence every term of

$$Q \equiv \frac{A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-1}^{n-2}, x_n^q)}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_n)}$$

is of the $(q-n+1)$th degree, as is illustrated in the examples of the preceding article. We shall now show that every possible term of the $(q-n+1)$th degree in the variables is found in Q, and that every such term is positive. That is to say, the quotient Q is the *complete symmetric function* of degree $(q-n+1)$ of $x_1, x_2, \cdots, x_n$.

Such a term of Q is

$$T \equiv x_1^3 x_2^0 x_3^6 \cdots x_{n-2}^7 x_{n-1}^2 x_n.$$

By successively applying **156**, we develop Q so that the terms containing x_n, $x_n x_{n-1}^2$, $x_n x_{n-1}^2 x_{n-2}^7$, etc., are at once distinguished. In the first place,

$$Q = \frac{A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-2}^{n-3}, x_{n-1}^{q-1})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_{n-1})} + \frac{x_n A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-2}^{n-3}, x_{n-1}^{q-2})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_{n-1})}$$

$$+ \frac{x_n^2 A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-2}^{n-3}, x_{n-1}^{q-3})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_{n-1})} + \cdots + \frac{x_n^{q-n} A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-2}^{n-3}, x_{n-1}^{n-1})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_{n-1})} + x_n^{q-n+1}.$$

The second term,

$$\frac{x_n A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-2}^{n-3}, x_{n-1}^{q-2})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_{n-1})} \equiv Q_1,$$

contains the first power of x_n; hence we must look for T in Q_1. Applying **156** to Q_1, as before, we have

$$Q_1 = x_n\left[\frac{A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-3}^{n-4}, x_{n-2}^{q-3})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_{n-2})} + \frac{x_{n-1}A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-3}^{n-4}, x_{n-2}^{q-4})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_{n-2})}\right.$$

$$+\frac{x_{n-1}^2 A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-3}^{n-4}, x_{n-2}^{q-5})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_{n-2})} + \cdots$$

$$\left.+\frac{x_{n-1}^{q-n-1} A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-2}^{n-2})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_{n-2})} + x_{n-1}^{q-n}\right].$$

In this expansion we must look for T in the third term

$$\frac{x_n x_{n-1}^2 A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-3}^{n-4}, x_{n-2}^{q-5})}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_{n-2})} \equiv Q_2.$$

Q_2 may be expanded as before; continuing in this way, we finally obtain the term

$$x_n x_{n-1}^2 x_{n-2}^7 \cdots x_3^6 \; \frac{A(x_1^0 x_2^4)}{\zeta^{\frac{1}{2}}(x_1 x_2)};$$

for the coefficient of $x_n x_{n-1}^2 x_{n-2}^7 \cdots x_3^6$ contains only x_1 and x_2, and is of the third degree. Upon performing the division, and multiplying, one of the terms is T. Since T is any term, the proposition is established.

Employing the notation H_r for the complete symmetric function of the rth degree, we may write the result of the present article

$$\frac{A(x_1^0, x_2^1, x_3^2, \cdots, x_{n-1}^{n-2}, x_n^q)}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_n)} = H_{q-n+1}(x_1, x_2, \cdots, x_n),$$

or simply $\qquad H_{q-n+1}$.*

For illustrations the student may refer to the examples in the preceding article.

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* With this notation, $H_0 = 1$, $H_{-r} = 0$.

Again,

$$\frac{\begin{vmatrix} 1 & x & x^2 & x^7 \\ 1 & y & y^2 & y^7 \\ 1 & z & z^2 & z^7 \\ 1 & t & t^2 & t^7 \end{vmatrix}}{\zeta^{\frac{1}{2}}(x, y, z, t)} = H_4 \equiv \Sigma x^4 + \Sigma x^3 y + \Sigma x^2 y^2 + \Sigma xyzt.$$

158. From the two preceding articles we have at once

$$H_r(x_1, x_2, \cdots, x_n) = x_n H_{r-1}(x_1, x_2, \cdots, x_n) + H_r(x_1, x_2, \cdots, x_{n-1}). \quad (1)$$

Whence we readily obtain

$$H_{r-1}(x_1, x_2, \cdots, x_{n+1}) = x_{n+1} H_{r-2}(x_1, x_2, \cdots, x_{n+1}) + H_{r-1}(x_1, x_2, \cdots, x_n);$$

$$\therefore H_{r-1}(x_1, x_2, \cdots, x_n) = H_{r-1}(x_1, x_2, \cdots, x_{n+1}) - x_{n+1} H_{r-2}(x_1, x_2, \cdots, x_{n+1}).$$

Substituting in (1),

$$H_r(x_1, x_2, \cdots, x_n) = x_n [H_{r-1}(x_1, x_2, \cdots, x_{n+1}) \\ - x_{n+1} H_{r-2}(x_1, x_2, \cdots, x_{n+1})] + H_r(x_1, x_2, \cdots, x_{n-1}). \quad (2)$$

Similarly,

$$H_r(x_1, x_2, \cdots, x_{n-1} x_{n+1}) = x_{n+1} [H_{r-1}(x_1, x_2, \cdots, x_{n+1}) \\ - x_n H_{r-2}(x_1, x_2, \cdots, x_{n+1})] + H_r(x_1, x_2, \cdots, x_{n-1}). \quad (3)$$

From (2) and (3),

$$H_r(x_1, x_2, \cdots x_n) - H_r(x_1, x_2, \cdots x_{n-1} x_{n+1}) \\ = (x_n - x_{n+1}) H_{r-1}(x_1, x_2, \cdots x_{n+1}). \quad (4)$$

159. If any alternant whose elements are powers (simple alternant) be divided by the difference product of its variables, the result is expressible as a determinant whose elements are complete symmetric functions of the variables. That is to say,

$$\frac{A(x_1^\alpha, x_2^\beta, \cdots x_n^\lambda)}{\zeta^{\frac{1}{2}}(x_1, x_2, \cdots x_n)} = \begin{vmatrix} H_\alpha & H_\beta & \cdots & H_\lambda \\ H_{\alpha-1} & H_{\beta-1} & \cdots & H_{\lambda-1} \\ \cdots & \cdots & \cdots & \cdots \\ H_{\alpha-n+1} & H_{\beta-n+1} & \cdots & H_{\lambda-n+1} \end{vmatrix}.$$

This may be proved as follows. For brevity we employ determinants of the third order, but the method applies, of course, to determinants of any order.* In the alternant

$$A(x_1^\alpha, x_2^\beta, x_3^\gamma) \equiv \begin{vmatrix} x_1^\alpha & x_1^\beta & x_1^\gamma \\ x_2^\alpha & x_2^\beta & x_2^\gamma \\ x_3^\alpha & x_3^\beta & x_3^\gamma \end{vmatrix} \tag{1}$$

subtract the first row from each of the other two, then remove the factors $(x_3 - x_1)$, $(x_2 - x_1)$. Afterward subtract the second row from the third, and remove the factor $x_3 - x_2$, employing equation (4), **158**. The result is

$$\frac{A(x_1^\alpha, x_2^\beta, x_3^\gamma)}{\zeta^{\frac{1}{2}}(x_1, x_2, x_3)} = \begin{vmatrix} H_\alpha(x_1) & H_\beta(x_1) & H_\gamma(x_1) \\ H_{\alpha-1}(x_1, x_2) & H_{\beta-1}(x_1, x_2) & H_{\gamma-1}(x_1, x_2) \\ H_{\alpha-2}(x_1, x_2, x_3) & H_{\beta-2}(x_1, x_2, x_3) & H_{\gamma-2}(x_1, x_2, x_3) \end{vmatrix}.$$

The determinant on the right we now transform as follows. Add the second row, multiplied by x_2, to the first, employing equation (1), **158**, and obtain the determinant

$$\begin{vmatrix} H_\alpha(x_1, x_2) & H_\beta(x_1, x_2) & H_\gamma(x_1, x_2) \\ H_{\alpha-1}(x_1, x_2) & H_{\beta-1}(x_1, x_2) & H_{\gamma-1}(x_1, x_2) \\ H_{\alpha-2}(x_1, x_2, x_3) & H_{\beta-2}(x_1, x_2, x_3) & H_{\gamma-2}(x_1, x_2, x_3) \end{vmatrix}.$$

Now add the third row, multiplied by x_3, to the second, again employing (1) of **158**; finally, add the second row, multiplied by x_3, to the first.

We then obtain

$$\frac{A(x_1^\alpha, x_2^\beta, x_3^\gamma)}{\zeta^{\frac{1}{2}}(x_1, x_2, x_3)} = \begin{vmatrix} H_\alpha(x_1, x_2, x_3) & H_\beta(x_1, x_2, x_3) & H_\gamma(x_1, x_2, x_3) \\ H_{\alpha-1}(x_1, x_2, x_3) & H_{\beta-1}(x_1, x_2, x_3) & H_{\gamma-1}(x_1, x_2, x_3) \\ H_{\alpha-2}(x_1, x_2, x_3) & H_{\beta-2}(x_1, x_2, x_3) & K_{\gamma-2}(x_1, x_2, x_3) \end{vmatrix},$$

as was to be shown. For an example,

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* The mode of proof here given is due to Mr. O. H. Mitchell, *American Journal of Mathematics*, Vol. IV., page 344.

$$\begin{vmatrix} a & a^4 & a^5 \\ b & b^4 & b^5 \\ c & c^4 & c^5 \end{vmatrix} = abc \begin{vmatrix} 1 & a^3 & a^4 \\ 1 & b^3 & b^4 \\ 1 & c^3 & c^4 \end{vmatrix}$$

$$= abc\,\zeta^{\frac{1}{2}}(a, b, c) \begin{vmatrix} H_0(a, b, c) & H_3(a, b, c) & H_4(a, b, c) \\ H_{-1} & H_2 & H_3 \\ H_{-2} & H_1 & H_2 \end{vmatrix}$$

$$= abc\,\zeta^{\frac{1}{2}}(a, b, c) \begin{vmatrix} H_2 & H_3 \\ H_1 & H_2 \end{vmatrix}$$

$$= abc\,\zeta^{\frac{1}{2}}(a, b, c) \begin{vmatrix} \Sigma a^2 + \Sigma ab & \Sigma a^3 + \Sigma a^2 b + \Sigma abc \\ \Sigma a & \Sigma a^2 + \Sigma ab \end{vmatrix}$$

$$= abc\,\zeta^{\frac{1}{2}}(a, b, c) \begin{vmatrix} -\Sigma ab & -\Sigma a^2 b - 2\,\Sigma\, abc \\ \Sigma a & \Sigma a^2 + \Sigma ab \end{vmatrix}$$

$$= abc\,\zeta^{\frac{1}{2}}(a, b, c) \begin{vmatrix} -\Sigma ab & \Sigma abc \\ \Sigma a & -\Sigma ab \end{vmatrix}$$

$$= abc\,\zeta^{\frac{1}{2}}(a, b, c)\,(\Sigma a^2 b^2 + \Sigma a^2 bc).$$

160. Form the product

$$P \equiv \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \times \begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & x_1 & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & x_2 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & x_n & 1 \end{vmatrix},$$

changing the columns of the first determinant into rows before multiplying. If we put

$$f_r(x) \equiv a_{1r}x^{n-1} + a_{2r}x^{n-2} + \cdots + a_{n-1\,r}x + a_{nr},$$

we find

$$P \equiv (-1)^{\frac{n}{2}(n-1)}\,|\,a_{1n}\,|\,\zeta^{\frac{1}{2}}(x_1, x_2, \cdots, x_n)$$

$$= \begin{vmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & f_2(x_2) & \cdots & f_n(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ f_1(x_n) & f_2(x_n) & \cdots & f_n(x_n) \end{vmatrix}.$$

If now

$$f_r(x_s) \equiv (x_s - y_r)^{n-1},$$

we must have

$$|a_{1n}| \equiv$$

$$\begin{vmatrix} 1 & \binom{n-1}{1}(-y_1) & \binom{n-1}{2}(-y_1)^2 & \binom{n-1}{3}(-y_1)^3 & \cdots & (-y_1)^{n-1} \\ 1 & \binom{n-1}{1}(-y_2) & \binom{n-1}{2}(-y_2)^2 & \binom{n-1}{3}(-y_2)^3 & \cdots & (-y_2)^{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \binom{n-1}{1}(-y_n) & \binom{n-1}{2}(-y_n)^2 & \binom{n-1}{3}(-y_n)^3 & \cdots & (-y_n)^{n-1} \end{vmatrix},$$

where

$$\binom{p}{q} \equiv \frac{p(p-1)(p-2)\cdots(p-q+1)}{q\,!}.$$

But this last determinant evidently equals

$$K(-1)^{\frac{n}{2}(n-1)}\zeta^{\frac{1}{2}}(y_1, y_2, y_3, \cdots, y_n),$$

where K is the product of all the binomial coefficients of order $n-1$. We have, accordingly,

$$\begin{vmatrix} (x_1-y_1)^{n-1} & (x_1-y_2)^{n-1} & \cdots & (x_1-y_n)^{n-1} \\ (x_2-y_1)^{n-1} & (x_2-y_2)^{n-1} & \cdots & (x_2-y_n)^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ (x_n-y_1)^{n-1} & (x_n-y_2)^{n-1} & \cdots & (x_n-y_n)^{n-1} \end{vmatrix}$$

$$= K\zeta^{\frac{1}{2}}(x_1, x_2, x_3, \cdots, x_n)\,\zeta^{\frac{1}{2}}(y_1, y_2, y_3, \cdots, y_n).$$

If now $x_r = y_r$, we have $\zeta(x_1, x_2, x_3, \cdots, x_n)$ in the form of a determinant.

161. Suppose now that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the roots of an equation

$$f(x) = 0. \tag{1}$$

Then $\zeta^{\frac{1}{2}}(a_1, a_2, a_3, \cdots, a_n)$ is the product of the differences of the roots of (1). Square this determinant, obtaining

$$\zeta(a_1, a_2, \cdots, a_n) = \begin{vmatrix} 1+1 \quad + \cdots + 1_n & a_1 + a_2 + \cdots + a_n & \cdots & a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} \\ a_1 \quad + a_2 \quad + \cdots + a_n & a_1^2 + a_2^2 + \cdots + a_n^2 & \cdots & a_1^n \quad + a_2^n \quad + \cdots + a_n^n \\ a_1^2 \quad + a_2^2 \quad + \cdots + a_n^2 & a_1^3 + a_2^3 + \cdots + a_n^3 & \cdots & a_1^{n+1} + a_2^{n+1} + \cdots + a_n^{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_1^{n-1} + a_2^{n-1} + \cdots + a_n^{n-1} & a_1^n + a_2^n + \cdots + a_n^n & \cdots & a_1^{2n-2} + a_2^{2n-2} + \cdots + a_n^{2n-2} \end{vmatrix}$$

$$= \begin{vmatrix} s_0 & s_1 & s_2 & \cdots & s_{n-1} \\ s_1 & s_2 & s_3 & \cdots & s_n \\ s_2 & s_3 & s_4 & \cdots & s_{n+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{n-1} & s_n & s_{n+1} & \cdots & s_{2n-2} \end{vmatrix},$$

where, as usual,

$$s_r \equiv a_1^r + a_2^r + \cdots + a_n^r.$$

162. The preceding article gives us an expression for the square of the differences of the roots in terms of s_i. We can also readily obtain an expression for the sum of the squares of the differences in terms of s_i as follows.

We have

$$\left\|\begin{matrix} 1 & 1 & 1 & \cdots & 1 \\ \alpha & \beta & \gamma & \cdots & \lambda \end{matrix}\right\|^2 = \begin{vmatrix} s_0 & s_1 \\ s_1 & s_2 \end{vmatrix} = \Sigma(\alpha - \beta)^2$$

by **58**.

163. We shall conclude our discussion of alternants with a theorem on the reduction of alternating functions to alternants.*

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* "Reduction of Alternating Functions to Alternants," Wm. Woolsey Johnson, *American Journal of Mathematics*, Vol. VII., page 345.

Any function of the form

$$\begin{vmatrix} \phi_1(a,\ bcd \cdots l) & \phi_2(a,\ bcd \cdots l) & \cdots & \phi_n(a,\ bcd \cdots l) \\ \phi_1(b,\ acd \cdots l) & \phi_2(b,\ acd \cdots l) & \cdots & \phi_n(b,\ acd \cdots l) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_1(l,\ abc \cdots k) & \phi_2(l,\ abc \cdots k) & \cdots & \phi_n(l,\ abc \cdots k) \end{vmatrix} \tag{1}$$

is evidently an alternating function of $a, b, c, \cdots l$, if

$$\phi(a,\ bcd \cdots l)$$

denotes a function of the n quantities $a, b, c, \cdots l$, which is symmetrical with respect to all the quantities except a. If each element of this determinant contains only the leading letter, (1) becomes

$$\begin{vmatrix} f_1(a) & f_2(a) & f_3(a) & \cdots & f_n(a) \\ f_1(b) & f_2(b) & f_3(b) & \cdots & f_n(b) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_1(l) & f_2(l) & f_3(l) & \cdots & f_n(l) \end{vmatrix}, \tag{2}$$

an alternant, which we represent, as usual, by its principal term,

$$[f_1(a), f_2(b), f_3(c), \cdots f_n(l)]. \tag{3}$$

Now, if the principal term of (1) can be separated into parts of the form (3), then the given alternating function (1) is equal to the sum of the alternants represented by these partial terms. This is proved as follows. Since an interchange of two rows of (1) is equivalent to an interchange of the corresponding letters, any term of (1) can be obtained from the principal term by a suitable transposition of the letters, and, similarly, the corresponding term in each of the alternants may be derived from its principal term by the same transposition of the letters; hence every term in the expansion of (1) is equal to the sum of the corresponding terms in the expansion of the alternants.

Accordingly, if a determinant of the form (1) is expressed, as usual, by writing its principal term in ( ), with commas between the elements, we may erase the commas, and treat the expression within the ( ) as an ordinary algebraic quantity.

Thus,

$$\begin{vmatrix} bcd & 1 & a & a^2 \\ cda & 1 & b & b^2 \\ dab & 1 & c & c^2 \\ abc & 1 & d & d^2 \end{vmatrix} \equiv A(bcd, 1, c, d^2) = A(a^0, b, c^2, d^3) = \zeta^{\frac{1}{2}}(a, b, c, d).$$

Again,

$$\begin{vmatrix} 1 & b^2+c^2 & a^2+bc \\ 1 & c^2+a^2 & b^2+ca \\ 1 & a^2+b^2 & c^2+ab \end{vmatrix} \equiv A(1,\ c^2+a^2,\ c^2+ab)$$

$$= A(a^0, b^0, c^4) + A(a^2, b^0, c^2) + A(a, b, c^2) + A(a^3, b, c^0)$$

$$= -A(a^0, b, c^3) = -(a+b+c)\,\zeta^{\frac{1}{2}}(a, b, c).$$

Functional Determinants.

164. Consider the following n functions of the n independent variables $x_1, x_2, \cdots x_n$.

$$\left.\begin{array}{l} y_1 = f_1(x_1, x_2, \cdots, x_n) \\ y_2 = f_2(x_1, x_2, \cdots, x_n) \\ \cdots \quad\quad \cdots \quad\quad \cdots \quad \cdots \\ y_n = f_n(x_1, x_2, \cdots, x_n) \end{array}\right\}. \qquad (1)$$

These functions will be independent if for every set of values of $y_1, y_2, \cdots y_n$ equations (1) determine one or more sets of values of $x_1, x_2, \cdots x_n$, so that these latter variables can in their turn be considered as functions of the n independent variables $y_1, y_2, \cdots y_n$.

Differentiating equations (1), we have

$$\left.\begin{aligned} dy_1 &= \frac{\delta f_1}{\delta x_1}dx_1 + \frac{\delta f_1}{\delta x_1}dx_2 + \cdots + \frac{\delta f_1}{\delta x_n}dx_n \\ dy_2 &= \frac{\delta f_2}{\delta x_1}dx_1 + \frac{\delta f_2}{\delta x_2}dx_2 + \cdots + \frac{\delta f_2}{\delta x_n}dx_n \\ \cdots & \quad \cdots \quad\quad \cdots \quad\quad \cdots \quad\quad \cdots \\ dy_n &= \frac{\delta f_n}{\delta x_1}dx_1 + \frac{\delta f_n}{\delta x_2}dx_2 + \cdots + \frac{\delta f_n}{\delta x_n}dx_n \end{aligned}\right\}. \tag{2}$$

Regarding equations (2) as a system of equations for determining $dx_1, dx_2, \cdots dx_n$, the determinant of this system

$$\begin{vmatrix} \frac{\delta y_1}{\delta x_1} & \frac{\delta y_1}{\delta x_2} & \cdots & \frac{\delta y_1}{\delta x_n} \\ \frac{\delta y_2}{\delta x_1} & \frac{\delta y_2}{\delta x_2} & \cdots & \frac{\delta y_2}{\delta x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\delta y_n}{\delta x_1} & \frac{\delta y_n}{\delta x_2} & \cdots & \frac{\delta y_n}{\delta x_n} \end{vmatrix} \equiv \frac{\delta(y_1, y_2, \cdots, y_n)}{\delta(x_1, x_2, \cdots, x_n)} \equiv J$$

is called the Jacobian of the given functions $y_1, y_2, \cdots y_n$. Or, in other words, the Jacobian of a set of n functions, each of n variables, is the determinant $|k_{1n}|$, in which the element k_{pq} is the first derivative of the pth function with respect to the qth variable. Thus, given

$$y_1 = az^2 + 2bzt + ct^2, \quad y_2 = a_1z^2 + 2b_1zt + c_1t^2.$$

The Jacobian

$$\frac{\delta(y_1, y_2)}{\delta(z, t)} = 4\begin{vmatrix} az + bt & bz + ct \\ a_1z + b_1t & b_1z + c_1t \end{vmatrix} = \frac{4}{z}\begin{vmatrix} y_1 & bz + ct \\ y_2 & b_1z + c_1t \end{vmatrix}$$

$$= \frac{4}{x}\begin{vmatrix} 0 & 0 & 1 \\ y_1 & bz + ct & c \\ y_2 & b_1z + c_1t & c_1 \end{vmatrix} = 4\begin{vmatrix} t^2 & -zt & z^2 \\ a & b & c \\ a_1 & b_1 & c_1 \end{vmatrix}.$$

165. If the functions $y_1, y_2, \cdots y_n$ are not independent, but are connected by a relation

$$\phi(y_1, y_2, \cdots y_n) = 0, \tag{3}$$

the Jacobian vanishes.

From (3) we have, by differentiating,

$$\left.\begin{array}{l} \dfrac{\delta\phi}{\delta y_1}\cdot\dfrac{\delta y_1}{\delta x_1} + \dfrac{\delta\phi}{\delta y_2}\cdot\dfrac{\delta y_2}{\delta x_1} + \cdots + \dfrac{\delta\phi}{\delta y_n}\cdot\dfrac{\delta y_n}{\delta x_1} = 0 \\ \dfrac{\delta\phi}{\delta y_1}\cdot\dfrac{\delta y_1}{\delta x_2} + \dfrac{\delta\phi}{\delta y_2}\cdot\dfrac{\delta y_2}{\delta x_2} + \cdots + \dfrac{\delta\phi}{\delta y_n}\cdot\dfrac{\delta y_n}{\delta x_2} = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \dfrac{\delta\phi}{\delta y_1}\cdot\dfrac{\delta y_1}{\delta x_n} + \dfrac{\delta\phi}{\delta y_2}\cdot\dfrac{\delta y_2}{\delta x_n} + \cdots + \dfrac{\delta\phi}{\delta y_n}\cdot\dfrac{\delta y_n}{\delta x_n} = 0 \end{array}\right\}. \tag{4}$$

From their mode of formation, equations (4) are simultaneous. Hence the determinant of the system vanishes by **77**; or,

$$J = 0.$$

We shall show presently that if the Jacobian of a set of functions vanishes, the functions are not independent.

166. The Jacobian of the implicit functions

$$\left.\begin{array}{l} F_1(x_1, x_2, \cdots x_n, y_1, y_2, \cdots y_n) = 0 \\ F_2(x_1, x_2, \cdots x_n, y_1, y_2, \cdots y_n) = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ F_n(x_1, x_2, \cdots x_n, y_1, y_2, \cdots y_n) = 0 \end{array}\right\} \tag{5}$$

is found as follows.

Equations (5) yield

$$-\frac{\delta F_i}{\delta x_k} = \frac{\delta F_i}{\delta y_1}\cdot\frac{\delta y_1}{\delta x_k} + \frac{\delta F_i}{\delta y_2}\cdot\frac{\delta y_2}{\delta x_k} + \cdots + \frac{\delta F_i}{\delta y_n}\cdot\frac{\delta y_n}{\delta x_k} \tag{6}$$

$$(i, k = 1, 2, \cdots, n).$$

Using equation (6), we find the product of

$$\begin{vmatrix} \frac{\delta F_1}{\delta y_1} & \frac{\delta F_1}{\delta y_2} & \cdots & \frac{\delta F_1}{\delta y_n} \\ \frac{\delta F_2}{\delta y_1} & \frac{\delta F_2}{\delta y_2} & \cdots & \frac{\delta F_2}{\delta y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\delta F_n}{\delta y_1} & \frac{\delta F_n}{\delta y_2} & \cdots & \frac{\delta F_n}{\delta y_n} \end{vmatrix} \text{ and } \begin{vmatrix} \frac{\delta y_1}{\delta x_1} & \frac{\delta y_2}{\delta x_1} & \cdots & \frac{\delta y_n}{\delta x_1} \\ \frac{\delta y_1}{\delta x_2} & \frac{\delta y_2}{\delta x_2} & \cdots & \frac{\delta y_n}{\delta x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\delta y_1}{\delta x_n} & \frac{\delta y_2}{\delta x_n} & \cdots & \frac{\delta y_n}{\delta x_n} \end{vmatrix}$$

to be

$$(-1)^n \begin{vmatrix} \frac{\delta F_1}{\delta x_1} & \frac{\delta F_1}{\delta x_2} & \cdots & \frac{\delta F_1}{\delta x_n} \\ \frac{\delta F_2}{\delta x_1} & \frac{\delta F_2}{\delta x_2} & \cdots & \frac{\delta F_2}{\delta x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\delta F_n}{\delta x_1} & \frac{\delta F_n}{\delta x_2} & \cdots & \frac{\delta F_n}{\delta x_n} \end{vmatrix}.$$

Whence

$$J \equiv \frac{\delta(y_1, y_2, \cdots y_n)}{\delta(x_1, x_2, \cdots x_n)} = (-1)^n \frac{\delta(F_1, F_2, \cdots F_n)}{\delta(x_1, x_2, \cdots x_n)} \div \frac{\delta(F_1, F_2, \cdots F_n)}{\delta(y_1, y_2, \cdots y_n)}. \tag{7}$$

If in (7) we put $n = 1$, we get

$$-\frac{\delta F_1}{\delta x_1} = \frac{\delta F_1}{\delta y_1} \frac{dy_1}{dx_1},$$

a well-known formula.

167. If in equations (5) we consider $x_1, x_2, \cdots x_n$ as functions of $y_1, y_2, \cdots y_n$, we obtain, as above,

$$(-1)^n \frac{\delta(F_1, F_2, \cdots F_n)}{\delta(y_1, y_2, \cdots y_n)} = \frac{\delta(F_1, F_2, \cdots F_n)}{\delta(x_1, x_2, \cdots x_n)} \times \frac{\delta(x_1, x_2, \cdots x_n)}{\delta(y_1, y_2, \cdots y_n)}. \tag{8}$$

From (7) and (8),

$$\frac{\delta(y_1, y_2, \cdots y_n)}{\delta(x_1, x_2, \cdots x_n)} \times \frac{\delta(x_1, x_2, \cdots x_n)}{\delta(y_1, y_2, \cdots y_n)} = 1. \tag{9}$$

168. Again, having given the $n+p$ functions,

$$\left.\begin{array}{r} F_1(x_1, x_2, \cdots x_n, y_1, y_2, \cdots y_{n+p}) = 0 \\ F_2(x_1, x_2, \cdots x_n, y_1, y_2, \cdots y_{n+p}) = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ F_{n+p}(x_1, x_2, \cdots x_n, y_1, y_2, \cdots y_{n+p}) = 0 \end{array}\right\}. \tag{10}$$

The Jacobian

$$J \equiv \frac{\delta(y_1, y_2, \cdots y_n)}{\delta(x_1, x_2, \cdots x_n)}$$

of the first n of these functions is found as follows. Differentiating equations (10), we find

$$-\frac{\delta F_i}{\delta x_k} = \frac{\delta F_i}{\delta y_1}\frac{\delta y_1}{\delta x_k} + \frac{\delta F_i}{\delta y_2}\frac{\delta y_2}{\delta x_k} + \cdots + \frac{\delta F_i}{\delta y_{n+p}}\frac{\delta y_{n+p}}{\delta x_k} \tag{b}$$

$$(i = 1, 2, \cdots n+p;\ k = 1, 2, \cdots n).$$

Now multiply together

$$\Delta \equiv \begin{vmatrix} \frac{\delta F_1}{\delta y_1} & \frac{\delta F_1}{\delta y_2} & \cdots & \frac{\delta F_1}{\delta y_{n+p}} \\ \frac{\delta F_2}{\delta y_1} & \frac{\delta F_2}{\delta y_2} & \cdots & \frac{\delta F_2}{\delta y_{n+p}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\delta F_{n+p}}{\delta y_1} & \frac{\delta F_{n+p}}{\delta y_2} & \cdots & \frac{\delta F_{n+p}}{\delta y_{n+p}} \end{vmatrix} \times \begin{vmatrix} \frac{\delta y_1}{\delta x_1} & \frac{\delta y_2}{\delta x_1} & \cdots & \frac{\delta y_n}{\delta x_1} \\ \frac{\delta y_1}{\delta x_2} & \frac{\delta y_2}{\delta x_2} & \cdots & \frac{\delta y_n}{\delta x_2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\delta y_1}{\delta x_n} & \frac{\delta y_2}{\delta x_n} & \cdots & \frac{\delta y_n}{\delta x_n} \end{vmatrix},$$

first writing J as a determinant of order $n+p$, thus :

$$\begin{vmatrix} \frac{\delta y_1}{\delta x_1} & \cdots & \frac{\delta y_n}{\delta x_1} & \frac{\delta y_{n+1}}{\delta x_1} & \cdots & \frac{\delta y_{n+p}}{\delta x_1} \\ \frac{\delta y_1}{\delta x_2} & \cdots & \frac{\delta y_n}{\delta x_2} & \frac{\delta y_{n+1}}{\delta x_2} & \cdots & \frac{\delta y_{n+p}}{\delta x_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\delta y_1}{\delta x_n} & \cdots & \frac{\delta y_n}{\delta x_n} & \frac{\delta y_{n+1}}{\delta x_n} & \cdots & \frac{\delta y_{n+p}}{\delta x_n} \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & 1_p \end{vmatrix}.$$

Calling the product P, we have

$$P \equiv (-1)^n \begin{vmatrix} \frac{\delta F_1}{\delta x_1} & \cdots & \frac{\delta F_1}{\delta x_n} & \frac{\delta F_1}{\delta y_{n+1}} & \cdots & \frac{\delta F_1}{\delta y_{n+p}} \\ \frac{\delta F_2}{\delta x_1} & \cdots & \frac{\delta F_2}{\delta x_n} & \frac{\delta F_2}{\delta y_{n+1}} & \cdots & \frac{\delta F_2}{\delta y_{n+p}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\delta F_{n+p}}{\delta x_1} & \cdots & \frac{\delta F_{n+p}}{\delta x_n} & \frac{\delta F_{n+p}}{\delta y_{n+1}} & \cdots & \frac{\delta F_{n+p}}{\delta y_{n+p}} \end{vmatrix}$$

$$\equiv (-1)^n \frac{\delta\,(F_1,\, F_2,\, \cdots\, F_{n+p})}{\delta\,(x_1,\, x_2,\, \cdots\, x_n,\; y_{n+1},\, y_{n+2},\, \cdots\, y_{n+p})};$$

since, by equation (b) for $k \leqq n$, the element a_{ik} of P is $-\frac{\delta F_i}{\delta x_k}$; and for $k > n$, $\frac{\delta F_i}{\delta y_k}$.

We have, accordingly,

$$J = \frac{P}{\Delta}.$$

169. Suppose equations (5) yield upon solution

$$y_1 = \phi_1(x_1,\, x_2,\, \cdots,\, x_n). \qquad (c)$$

Solve (c) for x_1, and substitute this value of x_1 in the remaining $n-1$ equations; then $y_2, y_3, \cdots y_n$ become functions of $y_1, x_2, \cdots x_n$. Thus

$$y_2 = \phi_2(y_1, x_2, \cdots x_n). \tag{d}$$

Solve (d) for x_2, substitute the result in the remaining $n-2$ equations; then $y_3, y_4, \cdots y_n$ become functions of

$$y_1, y_2, x_3, \cdots, x_n.$$

Thus
$$y_3 = \phi_3(y_1, y_2, x_3, \cdots, x_n). \tag{e}$$

Solve (e) for x_3, substitute as before; and so on.

We obtain the equations

$$\left.\begin{array}{ll} y_1 - \phi_1(x_1, x_2, \cdots x_n) & = 0 \\ y_2 - \phi_2(y_1, x_2, \cdots x_n) & = 0 \\ y_3 - \phi_3(y_1, y_2, x_3, \cdots x_n) & = 0 \\ \cdots \quad \cdots \quad \cdots & \cdots \\ y_n - \phi_n(y_1, y_2, \cdots y_{n-1}, \cdots x_n) & = 0 \end{array}\right\}. \tag{11}$$

By **166**,
$$J \equiv \frac{\delta(y_1, y_2, \cdots y_n)}{\delta(x_1, x_2, \cdots y_n)}$$

$$= (-1)^n$$

$$\begin{vmatrix} -\frac{\delta\phi_1}{\delta x_1} & -\frac{\delta\phi_1}{\delta x_2} & -\frac{\delta\phi_1}{\delta x_3} & \cdots & -\frac{\delta\phi_1}{\delta x_n} \\ 0 & -\frac{\delta\phi_2}{\delta x_2} & -\frac{\delta\phi_2}{\delta x_3} & \cdots & -\frac{\delta\phi_2}{\delta x_n} \\ 0 & 0 & -\frac{\delta\phi_3}{\delta x_3} & \cdots & -\frac{\delta\phi_3}{\delta x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & -\frac{\delta\phi_n}{\delta x_n} \end{vmatrix} \div \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\frac{\delta\phi_2}{\delta y_1} & 1 & 0 & \cdots & 0 \\ -\frac{\delta\phi_3}{\delta y_1} & -\frac{\delta\phi_3}{\delta y_2} & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 \\ -\frac{\delta\phi_n}{\delta y_1} & -\frac{\delta\phi_n}{\delta y_2} & -\frac{\delta\phi_n}{\delta y_3} & \cdots & 1 \end{vmatrix}$$

$$= \frac{\delta\phi_1}{\delta x_1} \cdot \frac{\delta\phi_2}{\delta x_2} \cdot \frac{\delta\phi_3}{\delta x_3} \cdots \frac{\delta\phi_n}{\delta x_n}.$$

That is to say, *the Jacobian of a set of functions* $y_1, y_2, \cdots y_n$, *each of* n *independent variables* $x_1, x_2, \cdots x_n$, *is expressible as a product of* n *differential coefficients of the functions* $\phi_1, \phi_2, \cdots \phi_n$, *where* ϕ_r *is a function of* $y_1, y_2, \cdots y_{r-1}, x_r, \cdots x_n$.

170. The result just obtained may be employed to show that *if the Jacobian of a set of functions vanishes, the functions are not independent.*

For, if
$$J \equiv \frac{\delta\phi_1}{\delta x_1} \cdot \frac{\delta\phi_2}{\delta x_2} \quad \cdots \quad \frac{\delta\phi_n}{\delta x_n}$$
vanishes, some one of the coefficients, say
$$\frac{\delta\phi_i}{\delta x_i} = 0,$$
where i has one of the values $1, 2, \cdots n$. But if $\dfrac{\delta\phi_i}{\delta x_i} = 0$, ϕ_i does not contain x_i, *i.e.*,
$$y_i = \phi_i(y_1, y_2, \cdots y_{i-1}, x_{i+1}, \cdots x_n).$$
Also
$$y_{i+1} = \phi_{i+1}(y_1, y_2, \cdots y_i, x_{i+1}, \cdots x_n).$$
From these two equations,
$$y_{i+1} = \psi_{i+1}(y_1, y_2, \cdots y_i, x_{i+2}, x_{i+3}, \cdots x_n);$$
therefore y_{i+1} does not contain x_{i+1}. In the same way we may show that y_{i+2} does not contain x_{i+2}, and so on. Hence, finally,
$$y_n = \psi_n(y_1, y_2, \cdots y_{n-1});$$
or y_n is expressible as a function of the remaining $n-1$ functions, and hence the given functions are not independent.

For example, if the given functions are

(1) $u = x + y$, (2) $v = x - z$, (3) $w = xy + xz - yz - z^2$,
$$J \equiv \begin{vmatrix} 1 & 1 & y+z \\ 1 & 0 & x-z \\ 0 & -1 & x-y-2z \end{vmatrix}.$$

J evidently vanishes. Accordingly, (1), (2), (3) are not independent. That the given functions are not independent is easily shown directly as follows. We readily obtain

$$y = u - v - z. \quad \therefore\ w = v(u - v),$$

as was to be shown.

171. If the functions $y_1, y_2, \cdots y_n$ are the n partial derivatives $\frac{\delta f}{\delta x_1}, \frac{\delta f}{\delta x_2}, \cdots \frac{\delta f}{\delta x_n}$, of a function $f(x_1, x_2, \cdots x_n)$, the Jacobian

$$H(f) \equiv \begin{vmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 \delta x_2} & \cdots & \frac{\delta^2 f}{\delta x_1 \delta x_n} \\ \frac{\delta^2 f}{\delta x_2 \delta x_1} & \frac{\delta^2 f}{\delta x_2^2} & \cdots & \frac{\delta^2 f}{\delta x_2 \delta x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\delta^2 f}{\delta x_n \delta x_1} & \frac{\delta^2 f}{\delta x_n \delta x_2} & \cdots & \frac{\delta^2 f}{\delta x_n^2} \end{vmatrix}$$

is called the Hessian of $(x_1, x_2, \cdots x_n)$. The Hessian is a symmetrical determinant, since

$$\frac{\delta^2 f}{\delta x_i \delta x_k} = \frac{\delta^2 f}{\delta x_k \delta x_i}.$$

If the derivatives $\frac{\delta f}{\delta x_1}, \frac{\delta f}{\delta x_2} \cdots \frac{\delta f}{\delta x_n}$, are connected by an equation, with constant coefficients

$$a_1 \frac{\delta f}{\delta x_1} + a_2 \frac{\delta f}{\delta x_2} + \cdots + a_n \frac{\delta f}{\delta x_n} = 0,$$

the Hessian must vanish.

172. Let $f_1, f_2, \cdots f_n$ be n given functions of the same variable x. Suppose the functions are connected by the linear relation

$$a_1 f_1 + a_2 f_2 + a_3 f_3 + \cdots + a_n f_n = 0, \tag{1}$$

in which $a_1, a_2, \cdots a_n$ are not functions of x. Differentiating (1) successively $n-1$ times, we have

$$\left.\begin{array}{l} a_1 f_1' \ \ + a_2 f_2' \ \ + a_3 f_3' \ \ + \cdots a_n f_n' \ \ = 0 \\ a_1 f_1'' \ + a_2 f_2'' \ + a_3 f_3'' \ + \cdots a_n f_n'' \ = 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_1 f_1^{n-1} + a_2 f_2^{n-1} + a_3 f_3^{n-1} + \cdots a_n f_n^{n-1} = 0 \end{array}\right\} \qquad (2)$$

Eliminating $a_1, a_2, \cdots a_n$ from (1) and (2), we find

$$\begin{vmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_1' & f_2' & f_3' & \cdots & f_n' \\ f_1'' & f_2'' & f_3'' & \cdots & f_n'' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_1^{n-1} & f_2^{n-1} & f_3^{n-1} & \cdots & f_n^{n-1} \end{vmatrix} \equiv D(f_1, f_2, f_3, \cdots f_n) = 0. \qquad (3)$$

The determinant of (3) has been called the *Wronskian* of $f_1, f_2, \cdots f_n$. We see from (3) that if the functions $f_1, f_2, \cdots f_n$ are connected by a linear equation of the form (1), the Wronskian vanishes.

173. If we denote the given functions by $y_1, y_2, \cdots y_n$, and the derivatives by $y_{11}, y_{21}, \cdots$ (*i.e.*, the second subscript denoting the derivatives), we may write (3)

$$\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_{11} & y_{21} & \cdots & y_{n\,1} \\ \cdots & \cdots & \cdots & \cdots \\ y_{1\,n-1} & y_{2\,n-1} & \cdots & y_{n\,n-1} \end{vmatrix} \equiv D(y_1, y_2, y_3, \cdots y_n) = 0. \qquad (4)$$

Now y being any function of x, we find

$$y^n D(y_1, y_2, y_3, \cdots y_n) = \begin{vmatrix} y_1 y & (y_1 y)_1 & \cdots & (y_1 y)_{n-1} \\ y_2 y & (y_2 y)_1 & \cdots & (y_2 y)_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ y_n y & (y_n y)_1 & \cdots & (y_n y)_{n-1} \end{vmatrix}, \qquad (5)$$

in which the subscript k of $(y_i y)_k$ means the kth derivative of $(y_i y)$. That is to say, the Wronskian of $y_1 y, y_2 y, \cdots y_n y$ is the product on the left in (5). This is made evident by noticing that since

$$(y_i y)_1 = y_{i1} y + y_i y', \qquad (y_i y)_2 = y_{i2} y + 2 y_{i1} y' + y_i y'',$$

etc., where $y', y'', \cdots$, are the successive derivatives of y, the determinant on the right becomes a sum of determinants, of which the first is the product on the left, and all the rest vanish.

174. We find

$$\frac{dD(y_1, y_2, \cdots, y_n)}{dx} = \begin{vmatrix} y_1 & y_{11} & \cdots & y_{1\,n-2} & y_{1\,n} \\ y_2 & y_{21} & \cdots & y_{2\,n-2} & y_{2\,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_n & y_{n\,1} & \cdots & y_{n\,n-2} & y_{n\,n} \end{vmatrix}; \qquad (A)$$

for in the sum of determinants which make up the derivative sought, all vanish except the one expressed in equation (A).

175. If in **173** we put $y = \dfrac{1}{y_1}$, the Wronskian on the right in (5) reduces to

$$\begin{vmatrix} \left(\dfrac{y_2}{y_1}\right)_1 & \left(\dfrac{y_2}{y_1}\right)_2 & \cdots & \left(\dfrac{y_2}{y_1}\right)_{n-1} \\ \left(\dfrac{y_3}{y_1}\right)_1 & \left(\dfrac{y_3}{y_1}\right)_2 & \cdots & \left(\dfrac{y_3}{y_1}\right)_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ \left(\dfrac{y_n}{y_1}\right)_1 & \left(\dfrac{y_n}{y_1}\right)_2 & \cdots & \left(\dfrac{y_n}{y_1}\right)_{n-1} \end{vmatrix} \equiv D\left[\left(\frac{y_2}{y_1}\right)_1, \left(\frac{y_3}{y_1}\right)_1, \cdots \left(\frac{y_n}{y_1}\right)_1\right].$$

Now

$$\left(\frac{y_2}{y_1}\right)_1 \equiv \frac{D(y_1, y_2)}{y_1^2}, \quad \left(\frac{y_3}{y_1}\right)_1 \equiv \frac{D(y_1, y_3)}{y_1^2}, \cdots \left(\frac{y_n}{y_1}\right)_1 \equiv \frac{D(y_1, y_n)}{y_1^2}.$$

Then if we put

$$D(y_1, y_2) \equiv z_2,\ D(y_1, y_3) \equiv z_3,\ \cdots\ D(y_1, y_n) \equiv z_n,$$

we get

$$D(y_1, y_2, \cdots y_n) = \frac{1}{y_1^{n-2}} D(z_2, z_3, \cdots z_n). \tag{6}$$

176. We shall employ the result just obtained to show that if the Wronskian of $y_1, y_2, \cdots y_n$ vanishes, the functions are connected by a linear equation having constant coefficients. Suppose that y_1 does not vanish, and since by hypothesis

$$D(y_1, y_2, \cdots y_n) = 0,$$

by (6) of the last article we must also have

$$\frac{1}{y_1^{n-2}} D(z_2, z_3, \cdots z_n) = 0.$$

Therefore, by **172**, the $n-1$ functions $z_2, z_3, \cdots z_n$ are connected by a linear relation, *i.e.*,

$$a_2 z_2 + a_3 z_3 + \cdots + a_n z_n = 0. \tag{7}$$

Dividing (7) by y_1^2, and restoring the values of $z_2, z_3, \cdots z_n$,

$$a_2\left(\frac{y_2}{y_1}\right)_1 + a_3\left(\frac{y_3}{y_1}\right)_1 + \cdots + a_n\left(\frac{y_n}{y_1}\right)_1 = 0. \tag{8}$$

Integrating (8), we find

$$a_1 y_1 + a_2 y_2 + a_3 y_3 + \cdots + a_n y_n = 0. \tag{9}$$

Therefore assuming that if the Wronskian of $n-1$ functions vanishes, the functions are connected by a linear relation, we have shown that when the Wronskian of n functions vanishes, the functions are connected by a linear relation. But the assumption is obviously true for two functions, hence the theorem is true universally.

Linear Substitution.

177. If the n functions (one or more)

$$\left.\begin{array}{l} f_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ f_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ f_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{array}\right\} \qquad (1)$$

are transformed into functions of $y_1, y_2, \cdots y_n$ by the following *linear substitutions*,*

$$\left.\begin{array}{l} x_1 = b_{11}y_1 + b_{12}y_2 + \cdots + b_{1n}y_n \\ x_2 = b_{21}y_1 + b_{22}y_2 + \cdots + b_{2n}y_n \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ x_n = b_{n1}y_1 + b_{n2}y_2 + \cdots + b_{nn}y_n \end{array}\right\}, \qquad (2)$$

the determinant $|b_{1n}|$ of the system (2) is called the *modulus of transformation*. If the modulus is unity, the substitution is unimodular. If $x_1, x_2, \cdots x_n$ are independent, the modulus cannot vanish.

178. If the functions (1) are transformed by means of (2) into

$$\left.\begin{array}{l} f_1 = m_{11}y_1 + m_{12}y_2 + \cdots + m_{1n}y_n \\ f_2 = m_{21}y_1 + m_{22}y_2 + \cdots + m_{2n}y_n \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ f_3 = m_{n1}y_1 + m_{n2}y_2 + \cdots + m_{nn}y_n \end{array}\right\}, \qquad (3)$$

the determinant of the system (3),

$$|m_{11} \; m_{22} \; \cdots \; m_{nn}|,$$

* The learner can understand the importance of linear substitution by noticing that such a substitution is the process involved in *transformation of coördinates* in Geometry.

equals the product of the determinant of the given system (1) by the modulus of transformation. That is to say,

$$|m_{1n}| = |a_{1n}| \times |b_{1n}|.$$

This is proved as follows. The coefficient of y_k,

$$m_{ik} \equiv a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk},$$

is found by multiplying equations (2) by $a_{i1}, a_{i2}, \cdots a_{in}$, respectively, and adding by columns. Whence, by **53**, we see that

$$\begin{vmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \times \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}.$$

179. If $f(x_1, x_2, \cdots x_n)$ is to be so transformed by the substitution

$$\left.\begin{array}{l} x_1 = \beta_{11} y_1 + \beta_{12} y_2 + \cdots + \beta_{1n} y_n \\ x_2 = \beta_{21} y_1 + \beta_{22} y_2 + \cdots + \beta_{2n} y_n \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ x_n = \beta_{n1} y_1 + \beta_{n2} y_2 + \cdots + \beta_{nn} y_n \end{array}\right\}, \qquad (a)$$

that $$y_1^2 + y_2^2 + \cdots + y_n^2 = x_1^2 + x_2^2 + \cdots + x_n^2,$$

the linear substitution is called *orthogonal*. The coefficients of an orthogonal substitution must satisfy the following conditions.

A. Since

$$\begin{aligned} & y_1^2 + y_2^2 + \cdots + y_n^2 \\ = & (\beta_{11} y_1 + \beta_{12} y_2 + \cdots + \beta_{1n} y_n)^2 + (\beta_{21} y_1 + \beta_{22} y_2 + \cdots + \beta_{2n} y_n)^2 \\ + & \cdots \quad \cdots \quad \cdots \quad \cdots \quad + (\beta_{n1} y_1 + \beta_{n2} y_2 + \cdots + \beta_{nn} y_n)^2 \\ = & (\beta_{11}^2 + \beta_{21}^2 + \cdots + \beta_{n1}^2) y_1^2 + (\beta_{12}^2 + \beta_{22}^2 + \cdots + \beta_{n2}^2) y_2^2 \\ + & \cdots + 2 y_1 y_2 (\beta_{11}\beta_{12} + \beta_{21}\beta_{22} + \cdots + \beta_{n1}\beta_{n2}) + \cdots, \end{aligned}$$

we must have

$$\text{I.}\quad \begin{cases} \beta_{1i}^2 + \beta_{2i}^2 + \cdots + \beta_{ni}^2 = 1 \\ \beta_{1i}\beta_{ik} + \beta_{2i}\beta_{2k} + \cdots + \beta_{ni}\beta_{nk} = 0 \quad (i, k = 1, 2, \cdots n). \end{cases}$$

B. If we wish to return to the original function from the transformed function, we must put

$$\text{II.}\qquad y_i = \beta_{1i}x_1 + \beta_{2i}x_2 + \cdots + \beta_{ni}x_n.$$

For from (a) we readily find

$$\beta_{1i}x_1 + \beta_{2i}x_2 + \cdots + \beta_{ni}x_n$$
$$= y_1(\beta_{11}\beta_{1i} + \beta_{21}\beta_{2i} + \cdots \beta_{n1}\beta_{ni}) + y_2(\beta_{12}\beta_{1i} + \beta_{22}\beta_{2i} + \cdots + \beta_{n2}\beta_{ni})$$
$$+ \cdots + y_n(\beta_{1n}\beta_{1i} + \beta_{2n}\beta_{2i} + \cdots + \beta_{nn}\beta_{ni}).$$

Now, by I., the coefficient of $y_i = 1$, and the other coefficients vanish.

C. The square of the determinant of the system (a) (modulus of transformation) is unity.

For

$$\text{III.}\qquad \begin{vmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n1} & \beta_{n2} & \cdots & \beta_{nn} \end{vmatrix}^2 = |\beta_{1n}|^2 \equiv |D_{1n}|.$$

$|D_{1n}|$ is a symmetrical determinant by **108**; since, by I.,

$$D_{ik} = 0, \quad D_{ii} = 1,$$

the truth of III. is obvious.

D. B_{ik} being the minor of β_{ik} in $|\beta_{1n}|$, we find

$$B_{ik} = \beta_{ik}|\beta_{1n}|.$$

For multiplying the equations

$$\begin{aligned} \beta_{11}\beta_{1k} + \cdots + \beta_{n1}\beta_{nk} &= 0, \\ \cdots \quad \cdots \quad \cdots \quad &\cdots, \\ \beta_{1k}\beta_{1k} + \cdots + \beta_{nk}\beta_{nk} &= 1, \\ \cdots \quad \cdots \quad \cdots \quad &\cdots, \\ \beta_{1n}\beta_{1k} + \cdots + \beta_{nn}\beta_{nk} &= 0, \end{aligned}$$

in order by $B_{i1}, B_{i2}, \cdots B_{in}$, and adding, we have

$$B_{ik} = \beta_{1k}(\beta_{11}B_{i1} + \cdots + \beta_{1n}B_{in}) + \cdots + \beta_{ik}(\beta_{i1}B_{i1} + \cdots + \beta_{in}B_{in}) + \cdots + \beta_{nk}(\beta_{n1}B_{i1} + \cdots + \beta_{nn}B_{in}).$$

But all the coefficients, except the coefficient of β_{ik}, vanish; hence

$$\text{IV.} \qquad B_{ik} = \beta_{ik} \,|\, \beta_{1n} \,|.$$

E. By the preceding condition IV.,

$$(\beta_{i1}\beta_{k1} + \cdots + \beta_{in}\beta_{kn}) \,|\, \beta_{1n} \,| = B_{i1}\beta_{k1} + \cdots + B_{in}\beta_{kn}.$$

The second member of this equation is $|\beta_{1n}|$, or 0, according as i and k are equal or unequal.

Whence

$$\text{V.} \quad \begin{cases} \beta_{i1}^2 + \beta_{i2}^2 + \cdots + \beta_{in}^2 = 1 \\ \beta_{i1}\beta_{k1} + \beta_{i2}\beta_{k2} + \cdots + \beta_{in}\beta_{kn} = 0. \end{cases}$$

F. The following relation holds between the minors of the modulus of the orthogonal substitution.

$$\text{VI.} \quad \begin{vmatrix} \beta_{r+1\,r+1} & \beta_{r+1\,r+2} & \cdots & \beta_{r+1\,n} \\ \beta_{r+2\,r+1} & \beta_{r+2\,r+2} & \cdots & \beta_{r+2\,n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n\,r+1} & \beta_{n\,r+2} & \cdots & \beta_{n\,n} \end{vmatrix} = |\,\beta_{1n}\,| \times \begin{vmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1\,r} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2\,r} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{r1} & \beta_{r2} & \cdots & \beta_{r\,r} \end{vmatrix}.$$

For, by **61**,

$$\begin{vmatrix} B_{11} & B_{12} & \cdots & B_{1\,r} \\ B_{21} & B_{22} & \cdots & B_{2\,r} \\ \cdots & \cdots & \cdots & \cdots \\ B_{r1} & B_{r2} & \cdots & B_{r\,r} \end{vmatrix} = |\,\beta_{1n}\,|^{r-1} \begin{vmatrix} \beta_{r+1\,r+1} & \beta_{r+1\,r+2} & \cdots & \beta_{r+1\,n} \\ \beta_{r+2\,r+1} & \beta_{r+2\,r+2} & \cdots & \beta_{r+2\,n} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{n\,r+1} & \beta_{n\,r+2} & \cdots & \beta_{n\,n} \end{vmatrix}.$$

Now, by IV.,

$$\begin{vmatrix} B_{11} & B_{12} & \cdots & B_{1r} \\ B_{21} & B_{22} & \cdots & B_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ B_{r1} & B_{r2} & \cdots & B_{rr} \end{vmatrix} = |\,\beta_{1n}\,|^r \times \begin{vmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1r} \\ \beta_{21} & \beta_{22} & \cdots & \beta_{2r} \\ \cdots & \cdots & \cdots & \cdots \\ \beta_{r1} & \beta_{r2} & \cdots & \beta_{rr} \end{vmatrix}.$$

Whence, equating the second members of these two equations, the relation VI. follows.

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